Fiber bundles, Yang and the geometry of spacetime.

A bachelor research in theoretical physics

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The entire course is hosted on YouTube at the following address:

www.youtube.com/playlist?list=PLPH7f_7ZlzxTi6kS4vCmv4ZKm9u8g5yic

We will be mainly interested in the fiber bundle formalism introduced.

Federico Pasinato

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Introduction

Since the early 20th century it has been understood that nature at the subatomic scale requires quantum mechanics, but the great breakthrough was the description of nature in terms of fields, led by Maxwell's ingenuous achievements of unifying Electricity and Magnetism with light. Einstein followed, placing Space and Time at the same footing relating them with matter through geometry. Herman Weyl tried to geometrize Maxwell's theory in the same spirit of Einstein and opened the doors to gauge theories, which C. N. Yang and R. Mills used to generalized the concept to non-abelian gauge groups to account for the other two forces of nature, the Weak and the Strong. It was then shown that such gauge theories should be regarded as specific geometric constructs, namely fiber bundle manifolds. The topic of research is to understand the mathematical formalism and how it relates the known forces of nature. Chapter one is intended as a historical introduction, setting the stage for future development, almost no equations and can be read by any person interested in science. Chapter two and three are intended for a graduating student in Physics, much of the mathematical core is just sketched with no mathematical rigor, just with physical intuitions. Chapter four assumes knowledge of the definitions, propositions, theorems and corollaries given in the appendices with the corresponding title, the author has tried to connect ideas from mathematics to physics through the fiber bundle formalism.

1 The first field theory

It is said that Coulomb, Gauss, Ampere and Faraday discovered 4 laws experimentally and James Clerk Maxwell wrote them into equations by adding the displacement current. Not entirely wrong but obscures the subtle interplay between geometrical and physical intuitions that were essential in the creation of field theory.

The first big step in the study of electricity was the invention by Volta (1745-1827) of the Voltaic Pile around 1800, a device of zinc and copper plates dipped in seawater brine. Then in Oersted (1777-1851) discovered in 1820 that an electric current would always cause magnetic needles in its neighborhood to move. This experiment lead to devices as the solenoid and to exact mathematical laws of Ampere (1775-1836) a learned mathematician that in 1827 worked out the exact magnetic forces in the neighborhood of a current as ″action at a distance″. Oersted's discovery also excited Faraday (1791-1867) that wrote to Ampere: "... *I am unfortunate in a want to mathematical knowledge and power of entering with facility any abstract reasoning. I am obliged to feel my way by facts placed closely together...* " (Sept. 3 1822). Without mathematical training and rejecting Ampere's action at a distance, Faraday used geometric intuition to "feel his way" in understanding his experiments. In 1831 he discovered electric induction and started 23 years long research, recording every experimental fact in his monumental collection <Experimental Research> without using a single formula. Faraday has discovered how to convert kinetic energy to electric energy and thereby how to make electric generators, but more important perhaps was his vague geometric conception of the "electro-tonic state" expressed in <ER> vol 3 p.443 he wrote: "... *a state of tension or a state of vibration or perhaps some other state analogous to the electric current, to which the magnetic forces are so intimately related.*" He believed that all metals take on the peculiar state of tension, appearing to be instantly assumed and he was perplexed by two facts: the magnet must be moved to produce induction and that induction often produce effects perpendicular to the cause. He was "feeling his way" in trying to understand electromagnetism, but today we have to "feel our way" in trying to understand his geometric intuition:

- Magnetic lines of forces (experimentally seen with sprinkling iron filings now called H, the magnetic field)
- Electrotonic state.

When in 1854 at 63 years of age, Faraday ceased his compilation of <ER> and the electro-tonic state remained an elusive geometric intuition, but along came J. C. Maxwell, a 23 years old graduated from Cambridge University that said: "... *I wish to attack Electricity*" . In two years Maxwell published the first of his three great paper which founded the Electromagnetic Theory as a Field Theory.

1.1 James Clerk Maxwell

Maxwell had learned Stokes's theorem and from reading Thomson's mathematical papers the usefulness of $H = curl\,(A)$ [1]. He figured out that Faraday's ″electro-tonic intensity″

is A, the vector potential. What Faraday had described in so many words was the equation $E = -deriv\,(A)$, that by taking the curl on both side yields the now known Faraday's law in differential form: $curl\,(E) = -deriv\,(H)$. . To avoid controversy with Thomson, Maxwell carefully wrote: "... *With respect to the history of the present theory, I may state that the recognition of certain mathematical functions as expressing the "electrotonic state" of Faraday, and the use of them in determining electrodynamic potentials and electromotive forces is, as far as I am aware, original; but the distinct conception of the possibility of the mathematical expression arose in my mind from the perusal of Prof. W. Thomson's papers...* "

In 1861 and 1862 Maxwell [2] added the displacement current and corrected his equations accounting for the effect due to the elasticity of the medium. He arrived at this correction according to his paper, through the study of a network of vortices, he had a geometrical model, that let him state: "*We can scarcely avoid inference that light consists in transverse undulation of the same medium which is the cause of electric and magnetic phenomena.*" . In other words, light is equal to electromagnetic waves. This is a momentous discovery of great importance for humankind. Any person, religious or not, must stop and wonder, because one of the greatest secrete of nature was revealed. He wrote a very clear exposition of the basic philosophy of field theory: "*In speaking of the Energy of the field, however, I wish to be understood literally. All energy is the same as mechanical energy, whether it exists in the form of motion or in that of elasticity, or in any other form. The energy in electromagnetic phenomena is mechanical energy. The only question is, where does it reside? On the old theories it resides in the electrified bodies, conducting circuits, and magnets, in the form of an unknown quality called potential energy, or the power of producing certain effects at a distance... on our theory it resides in the electromagnetic field, in the space surrounding the electrified and magnetic bodies, as well as in those bodies themselves, and is in two different forms, which may be described without hypothesis as magnetic polarization and electric polarization, or, according to a very probable hypothesis as the motion and the strain of one and the same medium.*" Maxwell still believed there had to be an "*aethereal medium*" .

The first important development in 20$^{\text{th}}$ century physicists' understanding of interactions was Einstein's 1905 special theory of relativity, according to which there is no "*aethereal medium*" , the electromagnetic field itself is the medium.

The vector potential A was first used by Lord Kelvin in 1851, who recognized its non-uniqueness, he recognized that by adding a gradient the laws of physics are invariant.

Throughout his life Maxwell always wrote his equations with the vector potential playing a key role, but after his death Heaviside and Hertz gleefully eliminated A :

$$\nabla \times \vec{E} = -\dot{\vec{H}} \qquad \nabla \times \vec{H} = 4\pi\vec{J} - \dot{\vec{E}} \qquad \nabla \cdot \vec{E} = 4\pi\rho \qquad \nabla \cdot \vec{H} = 0$$

Heaviside [3] said: "*The duplex method (referring to his recasting of the Faraday's law as curl E in parallel to the curl B for the Ampere's law as Maxwell has it) eminently suited for displaying Maxwell's theory and brings to light many useful relations which were* *formerly hidden from view by the intervention of the vector potential and its parasites(scalar potential)... in the present method we are from first to last, in contact with those quantities which are believed to have physical significance, instead of with mathematical functions of an essentially indeterminate nature, and with laws connecting them in the simplest form*" .

It was the advent of Quantum Mechanics that finalized the physical meaning of the vector potential and with the Aharonov-Bohm effect [4], there was no doubt, A cannot be eliminated and the flexibility in its definition is precisely the symmetry that determines the structure of the EM field. Thomson and Maxwell had both discussed what we now call the gauge freedom in $H = \nabla \times A$, but with development of QM, this freedom acquired additional meaning in physics.

1.2 Herman Weyl

Maxwell's equations have, beyond the Lorentz symmetry found by Einstein and Minkowski, another symmetry, namely Gauge Symmetry, discovered mainly by H. Weyl in the years 1919-1929. Weyl was one of the greatest mathematician of the 20th century, following his two predecessors Hilbert and Poincare, but he also focused on physics mostly in a philosophical way and the principal of gauge symmetry was a result of Weyl's excursion into the philosophy of Physics. Weyl's motivation was the geometrization of EM field, a challenge that Einstein put forward after he had geometrized gravity, in terms of curvature of spacetime, asking that EM field should also be geometrically understood.

Maxwell equations and QM were invariant under joint transformation $\psi \rightarrow \psi' = e^{i\alpha}\psi$ and $A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{e}\frac{\partial\alpha}{\partial x_\mu}$, the latter was known already, that is, from A_μ to A'_μ by adding a gradient such that the curl of A_μ remains invariant. But Weyl's intuition was transforming also ψ into ψ' by a phase factor, leading to QM invariant, calling them gauge transformation of first kind and gauge transformation of second kind respectively. He was until his death (1955) devoted to the idea of gauge invariance. He explicitly stated that the strongest argument for gauge theory was its relationship to charge conservation. Sadly his continued interest in the idea of gauge fields was not known among the physicists, he died without knowing that Yang and Mills had generalized his idea to non-Abelian Lie groups. Yang writes: "... had Weyl somehow come across our paper, I imagine he would have been pleased and excited, for we had put together two things that were very close to his heart: gauge invariance and non-Abelian Lie groups." . He was closed to non-Abelian group, because his paper of 1925 when he built upon Cartan's theory and developed the representation theory of Lie groups. Every Lie group has some representation in different dimensions, how to characterize them was one of the greatest contribution of Herman Weyl.

In 1919 Weyl [6] wrote: "... *the fundamental conception on which the development of Riemann's geometry must be based if it is to be in agreement with nature, is that of the infinitesimal parallel displacement vector... if in infinitesimal displacement of a vector, its direction keep changing then: Warum nicht auch seine Lange? (Why not also its length?)*". Weyl introduced a Streckenfacktor or Proportionalitatsfacktor: $exp\left(-\int eA\left(\nu\right)dx\hat{\ }\nu\right)/\gamma$, with γ being real. He identified the infinitesimal length changing factor, with the vector potential A at that point.

In 1925-1926 Fock and London independently pointed out that in Quantum Mechanics $(p - eA) \rightarrow -ih\left(\partial_\nu - \frac{ie}{h}A_\nu\right)$, which implies that Weyl's γ should be imaginary and in 1929 Weyl took this idea and published an important paper accepting that:

- A precise definition in QM of gauge transformation both for EM field and for wave function of charged particles, a coupled transformation on one hand gives the EM transformation and on the other hand it changes the phase of charged particles in QM.
- Maxwell's equations are invariant if one considers this combined gauge transformation.

Weyl's gauge invariance produced no new experimental results for more than 20 years, it was regarded as an elegant formalism but not essential, as C. N. Yang says, "*it seemed to be used only for a young theorist to spring a "smart" question at the end of a seminar: is your theory gauge invariant?*" . It was a mistake checking tool, because by starting with a gauge invariant theory, after all the calculations, the result must still be gauge invariant otherwise an error was made in the calculations.

1.3 Yang and Mills, the gauge symmetry

After World War 2 many new strange particles were found and the question of how they interact with each other emerged. This question let to a generalization of Weyl's gauge invariance to a possible new theory of interactions beyond EM. Thus, was born non-Abelian gauge theory. In 1954 C.N. Yang states [5]: "*... the electric charge serves as a source of electromagnetic field; an important concept in this case is gauge invariance which is closely connected with (1) the equation of motion of the electromagnetic field, (2) the existence of a current density, and (3) the possible interactions between a charged field and the electromagnetic field. We have tried to generalize this concept of gauge invariance to apply to isotopic spin conservation...*" . A concept originated in the 1930s from Heisenberg's and Wigner's papers. There was an additional conservation law, besides the electric charge conservation, such that if electric charge through gauge invariance can generate an EM field, wouldn't this conserved isotopic spin also generate a field?

In that year, gauge theory was generalized, Yang and Mills were motivated by:

- A principle of interaction, many new particles were discovered and the spin, parity and charge were accurately measured by the new experiments, the question arise, how do they interact?
- The electric charge serves as a source of electro-magnetic field; an important concept in this case is gauge invariance, relating the conservation of charge with the gauge transformation and that gives rise to the Maxwell's equations. Yang and Mills tried to generalize the concept of gauge invariance to apply to isotopic conservations, which is an empirical conservation law, so they asked: "here is a conservation law analogous, somehow similar, to the electric charge, should it also generate an interaction?"

- The usual principle of invariance under isotopic spin rotation is not consistent with the concept of localized fields.

At that time there was the concept of local field, Maxwell's equations, determined at every point independently, but the gauge transformation of Herman Weyl required an α which is constant, independent of spacetime and this concept is not in the spirit of field theory. With the above three motivations they converged on one generalization of Maxwell's equations. Since in ordinary EM, by making the gauge transformation of first and second kind as understood by Weyl, the result is form invariant, Yang and Mills tried a generalization to isotopic spin requiring the introduction of matrices resulting in ψ being now a two components object, a column matrix interpreting one entry for proton and one entry for neutron. Instead of the $A_\mu \to B_\mu$ as a two by two matrix. The next step was to copy Maxwell and find the field strength of the generalized theory $F_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu}$ except everything is a two by two matrices, but everything looked much more complicated and didn't go anywhere. After a while they tried to modify the equation, thinking that maybe the curl wasn't enough and they add a quadratic term:

$F_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} + ie\left(B_\mu B_\nu - B_\nu B_\mu\right)$

With this definition of $F_{\mu\nu}$ the gauge transformation becomes much simpler:

$F'_{\mu\nu} = B'_{\mu,\nu} - B'_{\nu,\mu} + ie\left(B'_\mu B'_\nu - B'_\nu B'_\mu\right) = S^{-1} F_{\mu\nu} S$

Later, they realized that from the mathematician point of view that quadratic term is necessary, but it was unknown to them in 1954 and this became the non-Abelian gauge theory. They immediately generalized to more complicated groups than the isotopic spin arising from an $SU\left(2\right)$ invariance, the procedure is the same for any Lie group. Once this generalization is understood, it is better instead of as matrices, to rewrite the equation in components form:

Maxwell's theory (Abelian)	**Non-Abelian Gauge Theory**
$F_{\mu\nu} = b_{\mu,\nu} - b_{\nu,\mu}$	$F^i_{\mu\nu} = b^i_{\mu,\nu} - b^i_{\nu,\mu} - c^i_{jk} b^j_\mu b^k_\nu$
$F_{\mu\nu,\nu} = -J_\mu$	$F^i_{\mu\nu,\nu} + c^i_{jk} b^j_\nu F^k_{\mu\nu} = -J^i_\mu$

Figure 1: Yang generalization of Maxwell

The first equations are a combination of Faraday's law and Gauss's law for Abelian and non-Abelian case. Similarly, the equations in the second row are a combination of Coulomb's and Ampere's laws in the in both cases.

The additional index i in the non-abelian case is there because when the group becomes larger than the electromagnetic group $U\left(1\right)$, there are more components, for example in the $SU\left(2\right)$ there are three components, $SU\left(3\right)$ there are eight components. Only in the special case of $U\left(1\right)$ it is possible to drop both, the index and the squared term, because in an Abelian case the commutator is zero. The term C^i_{jk} is called the structure constants, satisfying the following algebraic equation: $C^i_{ab} C^j_{ic} + C^i_{bc} C^j_{ia} + C^i_{ca} C^j_{ib} = 0$ known as Jacoby equation.

Group theory became important when Sophus Lie in the latter part of nineteen century generalized the theory into continuous groups, the groups of transformations. The distinction between different groups was done mainly by Elie Joseph Cartan, he solved the Jacoby equation, in 1895 he showed that there are four different types of solutions plus five exceptional type of solutions. In one stroke he clarified the structure of all Lie groups such as $SU(2), SU(3), G(2), \ldots$ are different type of solutions of this Lie algebra.

Another reason for which non-abelian gauge theory was criticized was that it seemed to require the existence of massless charged particles, Pauli believed that the theory would not make sense theoretically and experimentally. In 1960s the concept of spontaneous symmetry breaking was introduced which led to a series of major advances, finally to a $U(1) \times SU(2) \times SU(3)$ gauge theory of electroweak and strong interactions, called the Standard Model. In the forty some years since 1970 the international theoretical and experimental physics community working in "particles and fields" combined their efforts in the development and verification of this model, with spectacular success, climaxing in the discovery of the "Higgs Boson" in 2012 by two large experimental groups at CERN, each consisting of several thousand physicists. Despite its success, most physicists believe the standard model is not the final story. One of its chief ingredients, the symmetry breaking mechanism, was considered by Yang a phenomenological construct, but it remains a fundamental mechanism, provided that a scalr field exists with an associated potential. Analogies with the four ψ interaction in Fermi's beta decay theory can be made. That theory was also very successful for almost 40 years after 1933, but finally replaced by the deeper $U(1) \times SU(2)$ electroweak theory, that works around the problem of the operator being non-re-normalizable, generating non-unitary contributions at high energies.

1.4 The emergence of a central mathematical construct

Entirely independent of developments in physics there emerged, during the first half of the 20^{th} century, a mathematical theory called "*Fiber Bundle Theory*" , which had diverse conceptual origins:

- Differential forms (Cartan)
- Statistics (Hotelling)
- Topology (Whitney)
- Global differential geometry (Chern)
- Connection theory (Ehresman)
- Etc...

The great diversity of its conceptual origin indicates that fiber bundle is a central mathematical construct. It came as a great shock to both physicists and mathematicians when it became clear in the 1970s that the mathematics of gauge theory, both Abelian and non-Abelian, is exactly the same as that of fiber bundle theory. But it was a welcome shock, as it served to bring back the close relationship between the two disciplines which had been interrupted through the increasingly abstract nature of mathematics since the middle of the 20$^{\text{th}}$ century.

In 1975 after C.N. Yang learned the rudiments of fiber bundle theory from a mathematician colleague James Harry (Jim) Simons it was clear that Dirac, in the 1931 paper on the magnetic monopole, discovered trivial and non-trivial bundles before mathematicians. Ordinary EM uses the mathematics of trivial bundle, but EM with magnetic monopoles uses the full fiber bundle theory, which is the non-trivial one.

In the late 1960 Yang realizes the similarity between the equation of general relativity defining the Riemann curvature tensor and the non-Abelian field strength, in fact with a proper notation the two equations are not similar, they are exactly the same:

$$R^l_{ijk} = \frac{\partial}{\partial x^j}\begin{Bmatrix} l \\ ik \end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix} l \\ ij \end{Bmatrix} + \begin{Bmatrix} m \\ ik \end{Bmatrix}\begin{Bmatrix} l \\ mj \end{Bmatrix} - \begin{Bmatrix} m \\ ij \end{Bmatrix}\begin{Bmatrix} l \\ mk \end{Bmatrix}$$

$$F_{\mu\nu} = \frac{\partial B_\mu}{\partial x_\nu} - \frac{\partial B_\nu}{\partial x_\mu} + ie(B_\mu B_\nu - B_\nu B_\mu)$$

Figure 2: Taken from Yang's lecture notes - similarity between the non-abelian gauge theories

As mentioned before, Yang's colleague, the distinguished mathematician Jim Simon, looks at the equations and states: "*They must be both fiber bundles...* " . Yang was impressed with the fact that gauge fields are connections on fiber bundles which the mathematicians developed without reference to the physical world, and in 1975 speaking with Shiing-Shen Chern says: "*this is both thrilling and puzzling, since you mathematicians dreamed up these concepts out of nowhere.*" , he immediately protested, "*No, no. These concepts were not dreamed up. They were natural and real.*" . From the mathematician point of view, just building upon the concepts of beautiful structure, they have created the concept of fiber bundle. Then the fundamental question arises, since these natural and real ideas are without reference to the real world, why were they used in the real world?

In 1961, M. Stone [7] argues that: "*mathematics is entirely independent of the physical world, with no necessary connections beyond the vague and mystifying one implicit in the statement that thinking takes place in the brain.*" His reasoning was based on the fact that Mathematics in the beginning of 20th century became axiomatized, a tendency under Hilbert's influence assuming that mathematics can be made as a pure logical structure. In the 19th century Mathematics was deeply related to the physical world, but in the 20th century one of its greatest contribution was to break away from reality.

In 1972 F. J. Dyson [8] wrote: "*As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce.*" . Res Jost remarked on this "divorce" : "*As usual in such affairs, one of the two parties has clearly got the worst of it.*"

Mathematics and Physics remarried since the late 1970s, thanks to gauge theory and string theory. Wu and Yang played a central role by publishing the following table that got mathematicians again interested in physics:

Gauge field terminology	**Bundle terminology**
Gauge	Principal coordinate bundle
Gauge type	Principal fiber bundle
Gauge potential b_μ^j	Connection on a principal fiber bundle
S	Transition function
Phase factor	Parallel displacement
Field strength $F_{\mu\nu}^i$	Curvature
Source J_μ^i	**?**
EM	Connection on a trivial U(1) bundle
EM with magnetic monopoles	Connection on a non-trivial U(1) bundle
Isotopic spin gauge field	Connection on a SU(2) bundle
Dirac's monopole quantization	Classification of U(1) bundle according to first Chern class

Figure 3: Gauge field - Bundle relations

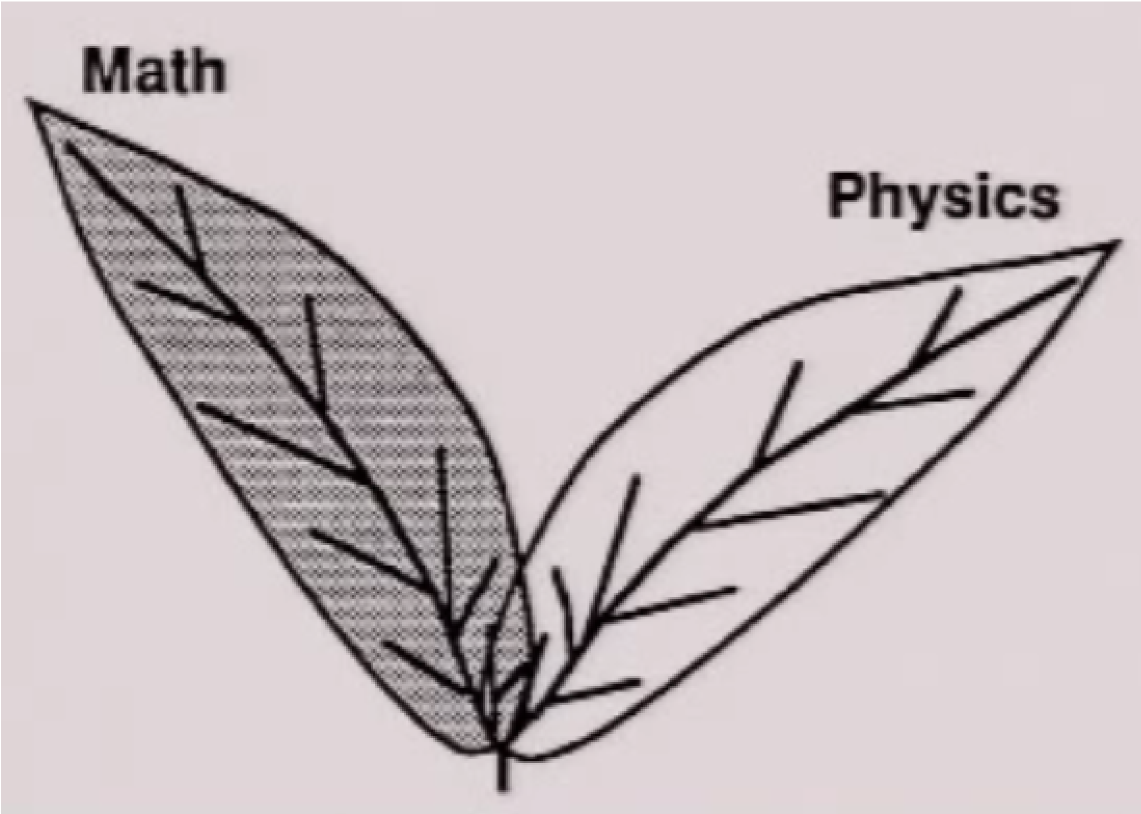


Figure 4: Relation between Math and Physics

Yang's point of view [9] is that Mathematics and Physics maybe viewed as two leaves, they mostly do not overlap, but in a little percentage of each domain there is a common ground and: "*the amazing thing is that in the overlapping region, ideas and concepts are shared in a deep way, but even there, the life force of each discipline runs along its own veins...they have their separate aims and tastes... they have different value judgments and they have different traditions.*" .

2 Abelian case

The concepts treated will be quantization, phase and symmetry. The main ideas are that in electromagnetism the vector potential A_μ is arbitrary to some extent $A_\mu \to A_\mu - \partial_\mu \Lambda$ and in quantum mechanics the phase of ψ is arbitrary $\psi \to \psi e^{i\theta}$. The question posed by Weyl is, can we make θ local, that is dependent on space and time $\theta = \theta(x, y, z, t)$?

Changing θ is like changing the phase convention, thinking about the complex plane, a phase convention is just a rotation. How to make precise the idea such that physics remains invariant by the choice of phase? Can it be changed at different spacetime points arbitrarily?

At the classical level, the principle of locality can be used to force electromagnetism and quantum mechanics together, as we will see, by starting with magnetostatics and by adding quantum mechanics it will be possible to have only a local phase.

2.1 Magnetostatics

Let $A \to A - \nabla\Lambda$ and $\psi \to \psi e^{i\theta}$, with a local phase depending on space $\theta = \theta(x, y, z)$, then by starting with EM there are four coupled equations for three unknowns: $\nabla \times B = 4\pi J$ and $\nabla \cdot B = 0$. The question is, what is physical? Because B has to little information, we have never seen magnetic monopoles, but the equations allow us to set a source or sink of magnetic field. Whilst A has too much information, it is important to go beyond B but not the full A. As was discussed earlier "a bit" of A will do and it is fundamental compared with B as the Aharonov-Bohm [4] experiment has shown. Let's arrive at the gauge transformation in EM. Assume that the magnetic field is given by the curl of the vector potential, then we can absorb the divergence free notion of the magnetic field with basic law from calculus: $\nabla \cdot B = 0 \Rightarrow B = \nabla \times A$. Since $A \to A - \nabla\Lambda$, by taking the curl on both side it yields: $\nabla \times A \to \nabla \times A - \nabla \times (\nabla\Lambda)$ which leaves the definition unchanged, since $\nabla \times (\nabla\Lambda) = 0$. The result is an improvement, because there are now three coupled equations for three unknowns since $\nabla \times (\nabla \times A) = 4\pi J$ can be rewritten as $\nabla(\nabla \cdot A) - \nabla^2 A = 4\pi J$, which yields for x -direction:

$$\partial_x (\partial_x A_x + \partial_y A_y + \partial_z A_z) - \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) A_x = 4\pi J_x$$

It is now possible to use the gauge degree of freedom, with $A \to A - \nabla\Lambda$, taking the divergence on both sides yield $\nabla A \to \nabla A - \nabla^2\Lambda$. By suitably choosing Λ , we can set $\nabla A - \nabla^2\Lambda = 0$. The trick is that we can always make such a choice by noting that $\nabla^2\Lambda = f$ has a solution. For physicists this is just like the Poisson equation in electrostatic, given a charge density find the potential, of course a solution exists and can be written down. Therefore, by suitable choice it is possible to go to a new A , whose divergence is always 0. This step achieves a great deal, now there are three decoupled equations in three unknowns since $\nabla(\nabla \cdot A) - \nabla^2 A = 4\pi J$ with the choice of Λ that yields $\nabla \cdot A = 0$, can be written for x -direction:

$$\nabla^2 A_x = -4\pi J_x$$

Therefore, for practical computations the advantage of using A reduces from 4 coupled to 3 coupled equations and the choice of gauge, the gauge degree of freedom, decouples the three equations. An analogy from electrostatic can be drawn: J_x "generates" A_x as ρ "generates" ϕ:

$$\nabla^2 \phi = -4\pi\rho$$

$$\nabla^2 A_x = -4\pi J_x$$

That yields as a solution: $A_x(r) = \int \frac{J_x\left(r'\right)}{|r-r'|} d^3 r'$. Gauge redundancy adds new unphysical degree of freedom in the A_μ description, and the gauge fixing bring us back to the correct counting of the physical degree of freedom in whatever description.

2.2 Adding Quantum Mechanics

In this subsection we will see that following Weyl's intuition we will find a relation between gauge degree of freedom in EM and the local phase degree of freedom in QM. In the following calculation the speed of light in vacuum $c = 1$. The quantum mechanical Schroedinger equation can be written as the Hamiltonian operator acting on a wave-function ψ yielding the eigenvalue E multiplying ψ. It is defined as $\left(\frac{1}{2m}(p - eA)^2 + V\right)\psi = E\psi$
Since $p = -i\hbar\nabla$, it is possible to define $(p - eA)\psi = (-i\hbar\nabla - eA)\psi$. The four-vector generalization is performed by letting the covariant exterior derivative be $D_\mu = \partial_\mu + eA_\mu$. The question is, if $A \rightarrow A - \nabla\Lambda$, does E stay the same? That is, a solution to the Schroedinger equation under a transformation of A, is it still a solution? So $|\psi|^2$ shouldn't change, therefore the only possibility is that the phase may change to compensate, in quantum mechanics we need this degree of freedom and the phase can be thought of as arbitrary. In the trivial example where the phase θ is constant, then a solution stays a solution, if $\psi \rightarrow \psi e^{i\theta}$, every term is multiplied by $e^{i\theta}$.

Let $\theta = \theta(x, y, z)$ be a function of position, then taking the gradient on both side of $\psi \rightarrow \psi e^{i\theta}$, yields: $\nabla\psi \rightarrow (\nabla\psi) e^{i\theta} + (i\nabla\theta)\psi e^{i\theta}$, which is not true that every factor gets multiplied by $e^{i\theta}$. The idea that will marry EM and QM comes about by doing both at the same time, changing the phase of the wave-function and changing A **:**

$$\psi \rightarrow \psi e^{i\theta}$$

$$A \rightarrow A - \nabla\Lambda$$

So that $(p - eA)\psi = (-i\hbar\nabla - eA)\psi$ **.** The idea is that if θ and Λ are suitably related, then the task is to find the proportionality constant and arrange for these two quantities $\nabla\theta$ and $\nabla\Lambda$ to cancel each other's.

$$(-i\hbar\nabla - eA)\,\psi \to -i\hbar\nabla\left(\psi e^{i\theta}\right) - e\,(A - \nabla\Lambda)\left(\psi e^{i\theta}\right)$$

$$-i\hbar\nabla\left(\psi e^{i\theta}\right) - e\,(A-\nabla\Lambda)\left(\psi e^{i\theta}\right) = (-i\hbar\nabla\psi)\left(e^{i\theta}\right) + (-i\hbar\psi)\,(i\nabla\theta)\left(e^{i\theta}\right) - eA\left(\psi e^{i\theta}\right) + e\nabla\Lambda\left(\psi e^{i\theta}\right)$$

The first and the third term is what we previously had $((-i\hbar\nabla - eA)\,\psi)\,e^{i\theta}$, so the second and fourth term must cancel, so that a solution stays a solution and it is independent of ψ:

$$(\hbar\nabla\theta + e\nabla\Lambda)\left(\psi e^{i\theta}\right) = 0$$

which is solved for $\Lambda = -\frac{\hbar}{e}\theta$. The electromagnetic function Λ and the quantum phase θ, if they are related in this way, with the proportionality factor involving the ratio $\frac{\hbar}{e}$, then a solution of the Schroedinger equation stays a solution and like in the trivial case, every term is just multiplied by $e^{i\theta}$. This solves the problem of the arbitrariness, to guarantee that we are describing the same physics, an A and an $A^{'}$ must only differ by $\frac{\hbar}{e}\nabla\theta$ such that a solution $\to$ solution:

$$\psi \to \psi e^{i\theta}$$

$$A \to A + \frac{\hbar}{e}\nabla\theta$$

In summary:

Gauge degree of freedom in EM $\Longleftrightarrow$ Local phase degree of freedom in QM

To turn it around, if we would have start with QM without knowing about EM, insisting that the phase degree of freedom must be made local, we would have discovered that we need A_μ, we are forced to introduce a four-vector potential $\to$ In some sense, forced to have EM to have a local phase symmetry. This is the essence of what professor Yang stated: "*symmetry dictates interactions*" . That is not how history went for EM, first we had the experimental knowledge, then we understood that the requirement of symmetry dictates interactions, but this is exactly what happened for Quantum Chromodynamics where the gauge transformation is a little bit more complicated. In this case it is this local invariance, that forces the interaction to be the Yang-Mills theory. The $U\,(1)$ symmetry forces the interaction for EM, so that for $\psi \to \psi e^{i\theta}$, we have θ as just a number. In Quantum Chromodynamics the $SU\,(3)$ is the symmetry group and to respect this local degree of freedom ψ transforms differently: $\psi \to \psi e^{i\theta^a T_a}$.

When we discuss quantum field theory, the concept of wave function is replaced by that of field and in the same spirit, but opposite view, we can take the Dirac Lagrangian in terms of a fermion field and imposing local U(1) invariance without knowing anything of QM.

2.3 Differential geometry for physicists

We will discuss a basic introduction to local vector space, parallel transport, covariant derivative, connection 1-form, basis vectors before going back to EM and the generalization to non-Abelian theories.

Local vector space The usual idea is that there is a complex-valued function ψ **,** which is a map from a domain to a range, that takes values from a base space B , to a vector space V .

Let $\psi(x) = \psi^1(x) + i\psi^2(x)$, with $x = x, y, z, t$.

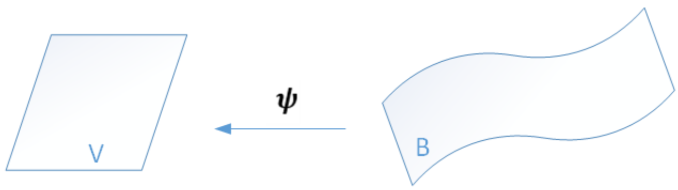


Figure 5: Idea of a wave function

From the above discussion about EM, we can choose a local phase convention in different places, that means is possible to rotate the wave-function at any point. It is assumed that there exists a "private" vector space unique to any point in the base space, let's say points x and y $\in B$, then there exist two vector spaces defined in the below picture by V(x) and V(y):

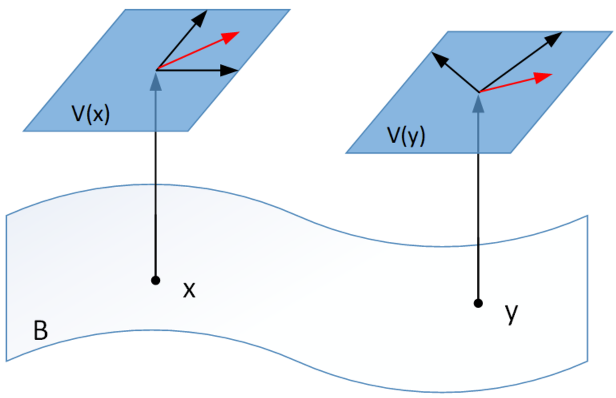


Figure 6: Local vector space

This is a pictorial representation of what is meant by the fact that the phase convention is local, the real and imaginary axes (black arrows) may rotate from a point x to a point y. Starting with the idea of a function, a mapping from one base space to one vector space, we want to generalize the idea of a function, as a mapping from one base space to many vector spaces. In the limiting case a "private" vector space at each point of the base space. The vector spaces are isomorphic, in the sense that they are not the same, but there exists a smooth transformation between them, they differ by an overall symmetric transformation and in particular, they can have different basis vectors. In general, $\psi(x) : B \rightarrow V(x)$, with:

B = b-dim manifold

V = v-dim manifold.

In the special case of EM: $v = 2$, $\psi = \psi^1 + i\psi^2 = e_1\psi^1 + e_2\psi^2$, with the basis vectors $e_1 = 1$ and $e_2 = i$.

Parallel transport When vectors live on different vector spaces, then the operation of subtracting vectors belonging to different vector spaces is not trivial, this is one of the key ingredients of relativity, since the law of motion involves the change in the momentum. The change in momentum, let's say a particle goes from "here" to "there" , means taking the difference between two momentum vectors which belong to two different vector spaces, one vector space at "here" and one vector space at "there" . To understand how to subtract the two vectors, we need the concept of parallel transport, that will keep the vector the same, but not its components, because the basis vectors may have changed.

Let $\psi(x) = A$, $\psi(x + \Delta x) = B$ and $A^{'}$ = the parallel transport of A at $x + \Delta x$ as in the below figure:

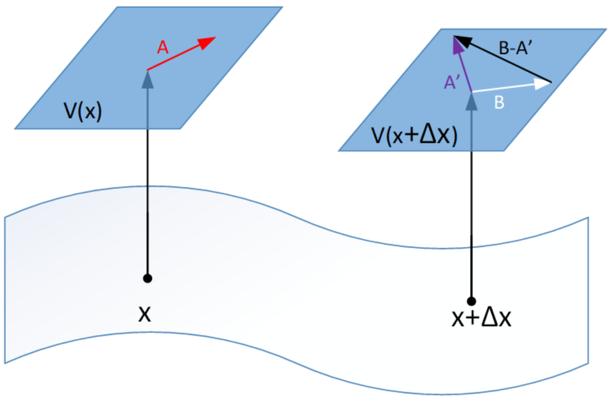


Figure 7: Parallel transport

Then the only subtraction that makes sense is $\psi(x + \Delta x) - \psi(x) = B - A'$. The physical motivation for this is the fact that the phase angle is arbitrary meaning that the vector spaces are twisted, rotated. In more generality, to push a vector by an amount Δx, the usual operator is the exponential defined as $e^{\Delta x^\mu \partial_\mu}$, that for small Δx (meaning infinitesimal transformation), can be expanded in a Taylor series and by keeping just the linear terms is given by: $e^{\Delta x^\mu \partial_\mu} \approx 1 + \Delta x^\mu \partial_\mu$.

Covariant derivative In a parallel transport it is also needed to do a linear transformation of the components, which differs from the identity also by a small amount proportional to Δx. Let the small transformation be defined as such $e^{\Delta x^{\mu}\Gamma_{\mu}}$, with $\Gamma = (v \times v)$ -matrix, that for small Δx can be also expanded in a Taylor series given by: $e^{\Delta x^{\mu}\Gamma_{\mu}} \approx 1 + \Delta x^{\mu}\Gamma_{\mu}$. To keep in mind is that the push (forward) of a single vector can be extended to the entire vector field. What is needed is a shift ∂_{μ}, component by component and a mixing (due to the rotation of axis). The mixing is the just described linear transformation of the components by Γ_{μ} . In the example of a quantum mechanical wave-function defined above on spacetime, Γ_{μ} is a set of 4 matrices, each of which is a (2×2) -matrix. That is in components form, Γ has three indices, one index μ running through the dimension of the base space and other two i, j labeling which row and which column of Γ : $(\Gamma_{\mu}\psi)^{i} = \Gamma^{i}_{\mu j}\psi^{j}$.

General relativity is a special example and the three indices are equivalent, but the distinction is still there, one index for the base space and two indices for the vector space. Convention, in dealing with real fields, Γ_{μ} is a real matrix, but if we need to preserve unitarity Γ_{μ} is an anti-Hermitian matrix.

Connection 1-form To relate infinitesimally closed vector spaces, mathematicians make use of Γ_{μ} to connect $V(x)$ to $V(x + \Delta x)$, calling it, the connection 1-form. The two operations needed to parallel transport a vector can be simplified $e^{\Delta x^{\mu}\partial_{\mu}} e^{\Delta x^{\mu}\Gamma_{\mu}} = e^{\Delta x^{\mu} D_{\mu}}$, by defining $D_{\mu} = \partial_{\mu} + \Gamma_{\mu}$ to be the covariant exterior derivative. Again, Γ_{μ} tells us how the basis in neighboring vector spaces are related, such that for any path in the base space, Γ_{μ} connect $V(x)$ for x along that path locally, then any point of the manifold can be reached, by summing up all the local paths.

This is the same as in the case of potential energy and work done, when calculating from an initial point to a final point, the work done is only defined locally, for infinitesimal displacement, and by adding them all up, it is possible to calculate from any initial to any final point. In general, there is no guarantee that the result is path independent and if the results are different for different paths, then you cannot define a potential energy, this is the same idea, but to understand the similarity, the concept of curvature is needed.

Basis vectors and connection Let $A(x)$ be a vector field, then define a new vector field displaced called $A_D(x)$ such that: $A_D(x) = A(x + \varrho) = A^{i}(x + \varrho)\, e_i(x + \varrho) \approx A(x) + \varrho^{\mu}\left(\left(\partial_{\mu}A^{i}\right) e_i + A^{j}(\partial_{\mu}e_j)\right)$ that means it is possible to expand $A(x + \varrho)$ in terms of the basis vectors $e_i(x + \varrho)$ and the expansion coefficients $A^{i}(x + \varrho)$, which are the components at the point $x + \varrho$ and then take the Taylor series expansion to first order. The key idea is that now we have to differentiate the basis vectors $\partial_{\mu}e_j$. If the basis vectors are constant, like in the case of a cartesian coordinate system in flat space or equivalently if there is just one vector space for every point of the base space, that is a global vector space, the second term $A^{j}(\partial_{\mu}e_j)$ drops out.

We can now identify, $D_{\mu}A = \left(\partial_{\mu}A^{i}\right) e_i + A^{j}(\partial_{\mu}e_j)$ and define $\partial_{\mu}e_j = P^{i}_{\mu j}e_i$ such that $D_{\mu}A = \left(\partial_{\mu}A^{i}\right) e_i + P^{i}_{\mu j}e_i A^{j} = \left(\partial_{\mu}A^{i} + P^{i}_{\mu j}A^{j}\right) e_i \Longrightarrow D_{\mu} = \partial_{\mu} + P_{\mu}$ which shows that P is indeed the connection and finally we can identify $\partial_{\mu}e_j = \Gamma^{i}_{\mu j}e_i$. The connection 1-form can be calculated if it is known how the basis vectors change with respect to position in base space.

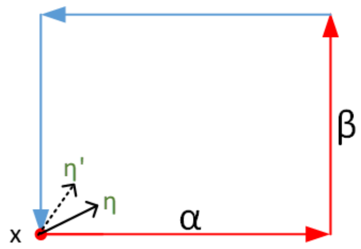


Figure 8: Closed loop parallel transport

Curvature If the connection Γ_μ is path independent, then all the $V(x)$ can be patched together consistently $V(x) \to V$; else there is curvature. Equivalently: Does a parallel transport around a closed loop reproduce the original vector? Let $A := \alpha^\mu D_\mu$ and $B := \beta^\mu D_\mu$ and the transformation to go around the loop symbolically as: $L = e^{-B}e^{-A}e^{B}e^{A}$, read from right to left. Meaning start from a point x, with a vector η, parallel transport an amount α, then displace an amount β, then $-\alpha$ and finally $-\beta$ to come back at the same point. If η' lies along η, then there is no curvature. As in the picture, if the dashed vector η', which is the original vector η after the displacement around the closed loop, points in another direction, then there is curvature. The transformation $L = e^{-B}e^{-A}e^{B}e^{A}$ around the loop can be written as: $L \approx 1 - [A, B] = 1 - \alpha^\mu \beta^\nu [D_\mu, D_\nu]$, which defines the curvature $R_{\mu\nu}$ in terms of the commutator between two covariant derivatives: $R_{\mu\nu} = [D_\mu, D_\nu]$. As we defined earlier, the covariant derivative is $D_\mu = \partial_\mu + \Gamma_\mu$, therefore we can rewrite the curvature in terms of the connection $R_{\mu\nu} = [D_\mu, D_\nu] = [\partial_\mu + \Gamma_\mu, \partial_\nu + \Gamma_\nu] = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu]$. This is a matrix equation, $(R_{\mu\nu}\psi)^i = R^i_{\mu\nu j}\psi^j$.

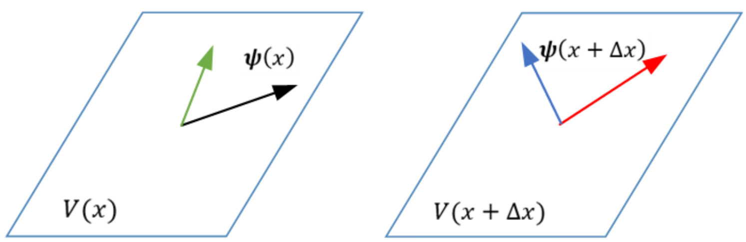


Figure 9: Write some caption here

Change of basis What if we change basis, what happens to the connection and the curvature? Let $\psi(x) \to e^{i\theta^a(x)T_a}\psi(x)$ by the following transformation $e^{i\theta^a(x+\Delta x)T_a}e^{-\Delta x^\mu D_\mu}e^{i\theta^a(x)T_a}$. Pictorially it means that we can make a local change of basis at $V(x)$, represented by a transformation from black to green arrow, whilst the change of basis at $V(x + \Delta x)$ is represented by going from the red to the blue vector. Since there exist a Γ that takes you from black arrow to red arrow. There must also exists a Γ' that represent directly a transformation in the changed basis, represented as a transformation from the green to the blue arrow.

Let $T(x) = \theta^a(x) T_a$ be an infinitesimal matrix, then

$$e^{i\theta^a(x+\Delta x)T_a} e^{-\Delta x^\mu D_\mu} e^{-i\theta^a(x)T_a} = e^{iT(x+\Delta x)} e^{-\Delta x^\mu D_\mu} e^{-iT(x)}$$

$$e^{iT(x+\Delta x)} e^{-\Delta x^\mu D_\mu} e^{-iT(x)} = (1 + i(T + \Delta x^\mu \partial_\mu T))(1 - \Delta x^\mu D_\mu)(1 - iT)$$

$$(1 + i(T + \Delta x^\mu \partial_\mu T))(1 - \Delta x^\mu D_\mu)(1 - iT) = 1 - \Delta x^\mu (D_\mu + i[T, D_\mu] - i\partial_\mu T)$$

Two things are infinitesimal, the displacement Δx^μ but also each rotation θ^a , because if it is possible to understand the change for an infinitesimal rotation it can be added up to a finite rotation. It is possible to define the new connection by understanding the covariant exterior derivative $D_\mu = \partial_\mu + \Gamma_\mu$ under this change of basis $D_\mu \to D_\mu + i[T, D_\mu] - i\partial_\mu T$, such that $\Gamma_\mu \to \Gamma_\mu + i[T, D_\mu] - i\partial_\mu T$ is the transformation law for the connection 1-form (a nonlinear transformation).

The transformation for the curvature it turns out it is easier than the transformation of the connection, it is purely linear if related to closed loops: $e^{iT} R_{\mu\nu} e^{-iT} = R_{\mu\nu} + i[T, R_{\mu\nu}]$. That is why the statement that $R_{\mu\nu}$ is equal to 0 , is an invariant statement, if true in one coordinate system, it remains true in all coordinate systems, which means that a vector under a parallel transport around a closed loop stays the same, a statement that can be made without reference to the coordinate system, it is not a statement about the components of the vector.

2.4 Back to EM

Armed with all the mathematical machinery, let's define the general form of the internal group to be $e^{i\theta^a T_a}$, with the set (T_a) as matrices running from 1 to the dimension of the internal space and the set (θ^a) as parameters. (Ex. For $SU(2)$, $a = 1, 2, 3.$) For EM all we are talking, is about $U(1)$, just a phase transformation, no need for an index a. Therefore T_a is just a 1x1-anti-Hermitian matrix (the identity) and $\theta^a = \theta$ the phase angle. In order to use Γ in the definition of the covariant exterior derivative D defined above, we can redefine the connection 1-form as $\Gamma_\mu = ieA_\mu$ and let it act on the complex valued wave-function: $\Gamma_\mu(\psi^1 + i\psi^2) = ieA_\mu(\psi^1 + i\psi^2)$. We can remove the complex imaginary " i " by introducing the matrix notation: $\Gamma_\mu \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = eA_\mu \varepsilon^i_j \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$. With $\varepsilon^i_j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is an anti-symmetric matrix which is representing just what a multiplication by i does, it turns real into imaginary and imaginary into minus real. The connection 1-form for the EM case can therefore be represented as ieA_μ or $eA_\mu \varepsilon^i_j$ and we can conclude that the vector potential tells you how to transport a wavefunction along a path. The two indices missing in the former representation are just in the factor i. We can now draw a new link between the two disciplines:

Mathematics:		Physics:
Connection 1-form Γ_μ	$\Longleftrightarrow$	Vector potential A_μ

The curvature in this case is easier, the group is Abelian and the connection along different direction commute, $[\Gamma_\mu, \Gamma_\nu] = 0$, then:

$$R_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu = ie\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = ieF_{\mu\nu}$$

Under a change of basis, let the infinitesimal transformation be $T\left(x\right) = \theta^a\left(x\right) T_a = \theta$, then connection changes as we defined by $\Gamma_\mu \to \Gamma_\mu + i\left[T, D_\mu\right] - i\partial_\mu T$, which for this case reduces to $A_\mu \to A_\mu - i\partial_\mu \theta$. We can now draw another new link between the two disciplines:

Mathematics:		Physics:
Connection 1-form Γ_μ	$\Longleftrightarrow$	Vector potential A_μ
Curvature 2-form $R_{\mu\nu}$	$\Longleftrightarrow$	EM field strength tensor $F_{\mu\nu}$

The electromagnetic field has to do with the fact that if the phase convention is changed at two different points and the two processes do not commute, then there is a non-trivial EM field. In the example above, if the complex planes fail to patch together trivially, then there is a non-trivial E and B. The curvature and the connection are the starting points in the discussion about Riemann Geometry and Fiber Bundles.

3 Non-Abelian case

The internal group has the general form $e^{i\theta^a T_a}$, the set $\{T_a\}$ are n by n matrices called generators and the set $\{\theta^a\}$ are parameters. The generators of a group are represented by matrices, which must be constructed with given properties for each given representation of the group (e.g. fundamental, adjoint representation.) For the general case $U(n)$, a runs from 1 to n^2, for the special case $SU(n)$, a runs from 1 to $n^2 - 1$. The wave function is replaced by field and can be thought of as a column vector $\psi = \left(\psi^1, \ldots, \psi^n\right)^T$. We can then define the connection 1-form analogously to the EM case by letting g be the coupling constant instead of the electric charge $e : \Gamma_\mu = igA^a_\mu T_a$. For example in the case of $SU(3)$, there are $n^2 - 1 = 8$ "photons", which in particle physics are called gluons. Another way of understanding is that now there are $a = n^2 - 1$ vector potentials A^a_μ, accompanied by the scalar potentials counterpart. Suppose you have an electron scattering off another electron, to represent that in field theory, we say it exchanges a photon and there is only one kind of photon, it couples only to charge. In Quantum Chromodynamics we have a quark scattering off another quark and the quarks have color, therefore the gluon that is exchanged must carry two indices so that color is conserved at any given vertex.

It is important to summarize that there are now three types of indices:

Index used	**What it refers to**	**An example**
$\mu = 1, \ldots, b$	Dim. of base space	A_μ
$i, j = 1, \ldots, v$	Dim. of vector space	ε^i_j
$a = 1, \ldots, n^2$	N. independent generators	T_a

The curvature in general is defined as $R_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] = igF^a_{\mu\nu} T_a$. Any matrix can be expanded in terms of T_a and the real expansion coefficients are $F^a_{\mu\nu}$. If $R_{\mu\nu}$ is defined as n by n traceless matrix, like in the case of $SU(n)$, the expansion does not involve the term T_0 .

The goal is to find an explicit expression for $F^a_{\mu\nu}$ by starting with the definition of the connection Γ_μ and curvature $R_{\mu\nu}$:

$$R_{\mu\nu} = \partial_\mu \left(igA^a_\nu T_a\right) - \partial_\nu \left(igA^a_\mu T_a\right) + (ig)^2 A^a_\mu A^b_\nu \left(T_a, T_b\right)$$

$$R_{\mu\nu} = ig\left(\left(\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) T_a + igA^a_\mu A^b_\nu f_{abc} T_c\right)\right)$$

.

Implying that $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf_{abc} A^a_\mu A^b_\nu$.

Four remarks about the field tensor:

- In EM, $F_{\mu\nu}$ has only two indices, both spacetime indices, but in the non-Abelian case $F^a_{\mu\nu}$, has the third index a in internal space.
- The strength is not given just by the curl of the vector potential, but there is another term which depends on the strength of the coupling constant.

- The combination eA_μ and gA_ν^a , always appears together, it is possible to shuffle coefficients, but separating the two, the charge and the vector potential, is a matter of convenience, the physical object is the combination.
- The field is nonlinear in the potential. If $F \sim \partial A + gAA$, then the Lagrangian schematically goes like the square of the field:
  - $L \sim F^2 \sim (\partial A + gAA)^2 \sim (\partial A)^2 + g(\partial A)A^2 + g^2A^4$, that in turns of Feynman's diagram respectively a propagation, a three-point interaction and a four-point interaction:

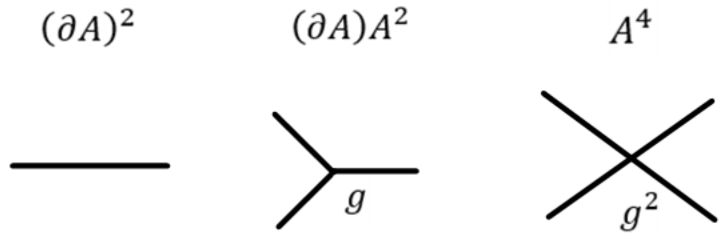


Figure 10: Feynman's diagram respectively a propagation, a three-point interaction and a four-point interaction

Therefore non-Abelian gauge theory is a non-linear interacting theory, the gluons can interact with each other and there is no possible principle of superposition.

3.1 Differential geometry for physicists - continued

This is intended to be another basic introduction to manifolds, coordinates, distance, metric, vectors, connection and curvatures to relate the non-Abelian case of General Relativity.

Manifolds and coordinates A manifold can be thought of as a set that is locally "similar" to flat space, Euclidean or Minkowski. On a manifold it is possible to define coordinates, they label the position in the manifold. The coordinates cannot be used to define distance or vectors. We are used to Cartesian coordinates that bundled together three concepts: (1) specification of the position of a point, (2) specification of distance between two points, (3) associated to vectors. On a curved manifold, it is fundamental to disentangle these three concepts. Coordinates only serve one purpose, specify the position of a point and they are defined with upper index, say x^μ . Since many coordinate systems are possible, we need to require that physics must be made independent of coordinates.

Distance and metric The central question is: how is distance, say Δs, related to a change in coordinates Δx^μ ? Riemannian geometry introduces the concept of metric to find distances, let two neighboring points say $P = \left(x^1, \ldots, x^n\right)$ and $Q = \left(x^1 + dx^1, \ldots, x^n + dx^n\right)$, then

$ds^2 = A\left(dx^1dx^1\right) + B\left(dx^1dx^2 + dx^2dx^1\right) + \ldots$, or in more general form:

$$ds^2 = g_{11}\left(dx^1dx^1\right) + g_{12}\left(dx^1dx^2\right) + g_{21}\left(dx^2dx^1\right) + \ldots + g_{nn}\left(dx^ndx^n\right)$$

or in compact form: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Which is a generalization of Pythagoras theorem to the following: (1) distance must still be quadratic, (2) need not be diagonal and (3) the coefficients need not to be constant, may depend on position. The tensor $g_{\cdot\cdot}$ with lower indices is called the metric an n by n matrix. The inverse matrix has upper indices $g^{\cdot\cdot}$ and it can be used to define new objects by exchanging the index from top to bottom or vice versa. For example: $a_\mu = g_{\mu\nu}a^\nu$ and $a^\mu = g^{\mu\nu}a_\nu$.

Vectors To define vectors, it is possible to start with the prototypical example, the displacement vector from two neighboring points and then define the other objects with the "same property" also as vectors. The problem is that on a curved space, the displacement is not what we would call a "straight arrow" , but an infinitesimal displacement we may argue that it is a "straight vector" . To recap: $x^\mu = \left(x^1, \ldots, x^n\right)$ is not a vector, the difference between two points $\Delta x^\mu = \left(\Delta x^1, \ldots, \Delta x^n\right)$ is not a vector, but the limit for infinitesimal displacement $dx^\mu = \left(dx^1, \ldots, dx^n\right)$ is a vector. The basis vectors at a point P , are defined as $dP = dx^\mu e_\mu$, with $e_\mu = \frac{\partial P}{\partial x^\mu}$. Which is just like:

$$i = e_x = \frac{\partial P}{\partial x} \; ; \qquad j = e_y = \frac{\partial P}{\partial y} \; ; \qquad k = e_z = \frac{\partial P}{\partial z} \; .$$

Tangent planes Now that we have defined vectors, we need to recognize that the displacement vector dx lives on a tangent plane at the point where the displacement is calculated. We can think that at every point of the manifold there exists a unique tangent plane and any displacement vector is a member, element, of a tangent plane. This is the key idea that links what we have discussed, at each tangent plane you can operate between vectors. When the manifold is curved, then the tangent planes are different at different points and we cannot operate with vectors living on different tangent planes. To be compared (subtracted, added. . . , anything learned from linear algebra), the vectors need to coexist on the same tangent plane. For example, on the surface of a sphere, boundary of a 3-dimensional ball, there is a unique "private" vector space at each point of the manifold. This is what Riemann answered, how to subtract two vectors in two different tangent planes and that is what Einstein used. First parallel transport to the same tangent plane and then compare. This is the analogous, in the case of EM when you transport a quantum mechanical wavefunction, of rotating the phase before you operate, like subtracting or taking derivatives. The idea is the same, in EM account for the change of phase, in GR account for the change of the vector's components before adding or subtracting.

Connection 1- form Since the tangent plane is a "private" vector space for each point of the manifold, to connect different vector spaces use the connection 1-form Γ_μ . As before, it connects $V(x)$ to $V(x + \Delta x)$, in Riemannian geometry it is called Christoffel symbol and as special case, the three indices describing it have the same dimension. Recalling that to parallel transport a vector, $e^{\Delta x^\mu \partial_\mu}$ shifts the argument and $e^{\Delta x^\mu \Gamma_\mu}$ causes the mixing of components, then $e^{\Delta x^\mu \Gamma_\mu} e^{\Delta x^\mu \partial_\mu} = e^{\Delta x^\mu D_\mu}$, with the covariant derivative $D_\mu = \partial_\mu + \Gamma_\mu$ implies that $\partial_\mu e_j = \Gamma^i_{\mu j} e_i$. The last equation tells how the basis vectors change with position, think that $\partial_\mu e_j$ is also a vector, then $\Gamma^i_{\mu j}$ are the expansion coefficients of the other basis vectors e_i .

In Riemannian geometry something special happens, the basis vectors are controlled by the metric, and one can prove the following $\Gamma^i_{\mu j} = \frac{1}{2} g^{\mu\lambda} \left(g_{\nu\lambda,\rho} + g_{\rho\lambda,\nu} - g_{\nu\rho,\lambda} \right)$ with the understanding that: $_{,\nu} := \left(\frac{\partial}{\partial x^\mu} \right)$. In the general case, the vector space has nothing to do with the underlying manifold and you cannot derive the connection in terms of the property of the manifold, but in the Riemannian case it is completely determined.

Curvature tensor As derived earlier, $R_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu]$, can also be written in component form resulting in $R^i_{j\mu\nu} = \partial_\mu \Gamma^i_{j\nu} - \partial_\nu \Gamma^i_{j\mu} + \Gamma^i_{\mu k} \Gamma^k_{\nu j} - \Gamma^i_{\nu k} \Gamma^k_{\mu j}$ up to minus sign, due to convention.

3.2 General Relativity

General relativity is the theory of gravity that replaces Newtonian inverse square law. In many ways it is similar to classical electromagnetism and there are in both theories two sides of the story. Consider the Lorentz law that tells you how charged particles move in an electric and magnetic field, that is given the two fields E and B, the force law gives you the acceleration allowing to predict the new velocity at a later time and therefore the new position can be derived. This is the first part, give the field, predict the motion of particles. The other half of the theory is the reverse, if you measure everything about the charged particles, where they are, how they are moving, in other words what is given are the charge and current densities, Maxwell's equations do predict the fields, namely given rho and J, the equations give you E and B.

In general relativity the situation is exactly the same, there are two sides of the story. The first part, analogous to Lorentz force law, is defined by first giving the field, specifying the metric of spacetime, then calculate how the particles move. Because the metric just tells you how to compute distances from coordinates, therefore suppose you know everything about g_{uv} then you can determine the motion of the particle:

$0 = dP = d\left(p^\mu \hat{e}_\mu\right) = \left(dp^\mu\right) \hat{e}_\mu + p^\mu \Gamma^\rho_{\mu\nu} dx^\nu \hat{e}_\rho$

$0 = (dp^\mu) \hat{e}_\mu + p^\rho \Gamma^\mu_{\rho\nu} dx^\nu \hat{e}_\mu$

$0 = \left(\mathrm{d}p^\mu + \ p^\rho \Gamma^\mu_{\rho\nu} d\ x^\nu\right) \hat{e}_\mu$

$0 = \mathrm{d}p^\mu + \ p^\rho \Gamma^\mu_{\rho\nu} dx^\nu$

$0 = \frac{\mathrm{d}p^\mu}{d\tau} + \frac{\Gamma^\mu_{\rho\nu} p^\rho d\ x^\nu}{d\tau}$

Let $p^\mu = m \frac{\mathrm{d}x^\mu}{d\tau}$ then $\frac{d^2 x^\mu}{d\tau} = -\Gamma^\mu_{\rho\nu} \frac{dx^\nu}{d\tau} \frac{dx^\nu}{d\tau}$ is the equation of motion for a point particle.

The second part is again similar, but harder and it will not be derived. If you know everything about the particles, where they are and how they are moving, which is the energy momentum densities and energy momentum flux, which together form the energy momentum current densities tensor, (that is "similar" to the charge densities and current densities in the EM case), which is the source, then you can find the field g_{uv}. This, like Maxwell's equations counterpart for EM, is the harder part. G_{uv} is a symmetric tensor, unlike A_u which has only 4 quantities, it has 10 quantities that implies 10 coupled equations in 10 unknowns. Unlikely EM, where it was possible with a gauge choice to uncouple the equations, now it is not "yet" possible to decouple them.

3.3 Concluding remarks, a path to unification

Letting non-Abelian gauge theory aside for the moment, what has been described is a formal resemblance between EM and curved spacetime (gravity) with a deep formal analogy in terms of the mathematics, the question may be, is there a deep physical unification behind this formal analogy? The reason that they are described by similar mathematical constructs may suggest that there is a formal deep relationship between the two. This is one of the path towards unification, looking at EM and GR in terms of connections and curvatures. Of course, an undergraduate student may look at Coulomb's and Newton's law, and the mathematical similarity and think about the relationship between electricity and gravity, is there a common basis for the two? A graduate student will of course say no, electricity is governed by Maxwell's equations that are related to the quantum mechanical phase, which substituted Coulomb's law and gravity is understood with Einstein's theory which has Newton's law as an approximation. It seems to the author that even though we changed and better understood both theories, the mathematical similarity just got even deeper, both in terms of connections and curvatures. By looking at the equations of motion for EM and Gravity there are already clear distinctions between the two, the equations of motion in the EM case are linear in the velocity and governed by the change of curvature which is related to the sources and currents, whilst for gravity, the equations of motion are quadratic in the velocity and given directly by the connection one-form. Can we pursue an analogy? One possible solution is trying to look at GR with one more derivative and try to assign meaning to the derivative of the energy momentum densities as the actual sources and currents responsible for the change of curvature of spacetime, but in the literature it seems that one is running into a wall.

4 Fiber bundle

This chapter follows form the mathematical definitions, propositions, theorems and proofs that can be found in the Appendices.

4.1 Topological manifolds and bundles

Intuitively, a d-dimensional manifold is a topological space which locally (i.e. around each point) looks like $\mathbb{R}^d$. Note that, strictly speaking, what is defined in the appendix are *real* topological manifolds. We could define *complex* topological manifolds as well, simply by requiring that the map x be a homeomorphism onto an open subset of $\mathbb{C}^d$.

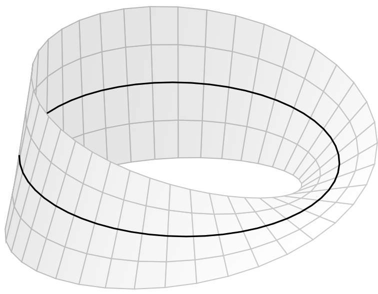
Example 4.1. Trivially, $\mathbb{R}^d$ is a d-dimensional manifold for any $d \geq 1$. The space S^1 is a 1-dimensional manifold while the spaces S^2, C and T^2 are 2-dimensional manifolds.

Example 4.2. The space S^1 is a submanifold of $\mathbb{R}^2$ while the spaces S^2, C and T^2 are submanifolds of $\mathbb{R}^3$.

Example 4.3. The cylinder $C = S^1 \times \mathbb{R}$ is a 2-dimensional manifold.

4.1.1 Bundles

Products are very useful. Very often in physics one intuitively thinks of the product of two manifolds as attaching a copy of the second manifold to each point of the first. However, not all interesting manifolds can be understood as products of manifolds. A classic example of this is the *Möbius strip*.[1]

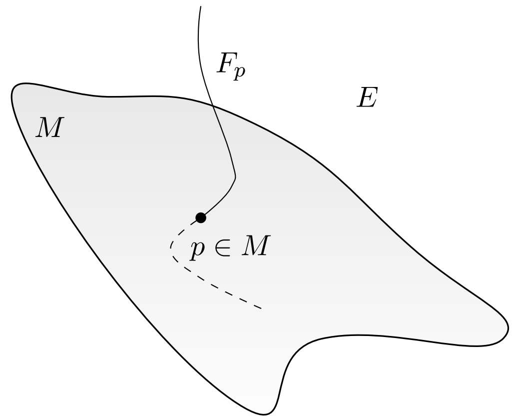


It looks locally like the finite cylinder $S^1 \times [0, 1]$, which we can picture as the circle S^1 (the thicker line in figure) with the finite interval $[0, 1]$ attached to each of its points in a "smooth" way. The Möbius strip has a "twist", which makes it globally different from the cylinder.

Let denote the bundle (E, π, M) by $E \xrightarrow{\pi} M$.

Intuitively, the fibre at the point $p \in M$ is a set of points in E (represented below as a line) attached to the point p. The projection map sends all the points in the fibre F_p to the point p.

[1]The TikZ code for the Möbius strip was written by Jake on TeX.SE.



4.1.2 Product bundles

Example 4.4. A trivial example of a bundle is the *product bundle*. Let M and N be manifolds. Then, the triple $(M \times N, \pi, M)$, where:

$$\begin{aligned} \pi \colon M \times N &\to M \\ (p, q) &\mapsto p \end{aligned}$$

is a bundle since (one can easily check) π is a continuous and surjective map. Similarly, $(M \times N, \pi, N)$ with the appropriate π, is also a bundle.

4.1.3 Fiber Bundles

Example 4.5. In a bundle, different points of the base manifold may have (topologically) different fibres. For example, consider the bundle $E \xrightarrow{\pi} \mathbb{R}$ where:

$$F_p := \operatorname{preim}_\pi(\{p\}) \cong_{\text{top}} \begin{cases} S^1 & \text{if } p < 0 \\ \{p\} & \text{if } p = 0 \\ [0,1] & \text{if } p > 0 \end{cases}$$

Example 4.6. The bundle $M \times N \xrightarrow{\pi} M$ is a fibre bundle with fibre $F := N$.

Example 4.7. The Möbius strip is a fibre bundle $E \xrightarrow{\pi} S^1$, with fibre $F := [0, 1]$, where $E \neq S^1 \times [0, 1]$, i.e. the Möbius strip is not a product bundle.

Example 4.8. A $\mathbb{C}$-line bundle over M is the fibre bundle (E, π, M) with fibre $\mathbb{C}$. Note that the product bundle $(M \times \mathbb{C}, \pi, M)$ is a $\mathbb{C}$-line bundle over M, but a $\mathbb{C}$-line bundle over M need not be a product bundle.

Intuitively, a section is a map σ which sends each point $p \in M$ to *some* point $\sigma(p)$ in its fibre F_p, so that the projection map π takes $\sigma(p) \in F_p \subseteq E$ back to the point $p \in M$.

Example 4.9. Let $(M \times F, \pi, M)$ be a product bundle. Then, a section of this bundle is a map:

$$\begin{aligned} \sigma \colon M &\to M \times F \\ p &\mapsto (p, s(p)) \end{aligned}$$

where $s\colon M \to F$ is any map.

The structure-preserving maps for bundles are called bundle isomorphisms.

Example 4.10. The cylinder C is trivial as a bundle, and hence also locally trivial.

Example 4.11. The Möbious strip is not trivial but it is locally trivial.

From now on, we will mostly consider locally trivial bundles.

Remark 4.12. In quantum mechanics, what is usually called a "wave function" is not a function at all, but rather a section of a $\mathbb{C}$-line bundle over physical space. However, if we assume that the $\mathbb{C}$-line bundle under consideration is locally trivial, then each section of the bundle can be represented (locally) by a map from the base space to the total space and hence it is appropriate to use the term "wave *function*".

4.1.4 Application

The author would like the reader to stop and wonder about the way spacetime itself is defined. Einstein put at the same footing both, space and time realizing them into a unique 4-dimensional manifold. But for anyone not in the field of physics, the concept of space is indeed different from the concept of time: "...glazing at our surrounding and even up at the stars, anybody can dream about the infinity of space, but anyone living will realize the finiteness of time...". Assuming that space and time are two different topological manifolds that can be described as one unique product bundle manifold is too restricted. A more general choice which allows locally to be still represented as product bundle, but doesn't make any assumption on the global feature, is to interpret spacetime as a fiber bundle. One approach is to let the base manifold to represent time and the fiber at each point of the base manifold to represent space. This isn't such a different concept from what we already assume: "At every instant of time, it exists a space, that space is not the same at the next instant of time, further more at every point of that space, a projection to the base manifold will yield the same instant of time." In a even more down to Earth example, every person that lives on this planet close to a meridian connecting the south pole to the north pole will experience the same time, therefore a clock on their wrist would be like a projection from total space down to base space. By allowing this definition of spacetime, we need to consider not the functions of time defining the position of an observer, but rather sections. The initial conditions that are always used in every physical example to set the position in space of an observer as functions of time, maybe looked instead as sections, because by using these choices, the smooth maps chosen in a local domain of the base manifold, will induce a local trivialization allowing to go back to the original definition of spacetime as product of the two manifolds of space and of time. There is no need to be limited here, we can make the base manifold complex and letting time be just the real part, allowing for the imaginary part to be orthogonal to time and no preferred direction, like a phase. We can even turn the problem around and move towards another definition of spacetime, by letting the base manifold to be just space and the real valued fiber at each point of space to represent time itself. What is achieved in this more general thinking is to be more humble, assuming that spacetime is just a product bundle locally makes perfect sense and we can make measurements on it, but believing that the structure is preserved at large scale (like the one used for Cosmology) or at the small scale (like the one used for Quantum Mechanics) is requiring too much from our local point of view as meter sized observers of our universe.

4.2 Principal and Associated bundle

We would like to define a vector field on a manifold M as a "smooth" map that assigns to each $p \in M$ a tangent vector in T_pM. However, since this would then be a "map" to a different space at each point, it is unclear how to define its smoothness. The simplest solution is to merge all the tangent spaces into a unique set and equip it with a smooth structure, so that we can then define a vector field as a smooth map between smooth manifolds.

4.2.1 Tangent bundle

Let $\mathscr{A}_M$ be a smooth atlas on M and let $(U,x) \in \mathscr{A}_M$. If $X \in \mathrm{preim}_\pi(U) \subseteq TM$, then $X \in T_{\pi(X)}M$, by definition of π. Moreover, since $\pi(X) \in U$, we can expand X in terms of the basis induced by the chart (U,x):

$$X = X^a \left(\frac{\partial}{\partial x^a}\right)_{\pi(X)},$$

where $X^1, \ldots, X^{\dim M} \in \mathbb{R}$. We can then define the map

$$\begin{aligned} \xi \colon \mathrm{preim}_\pi(U) &\to x(U) \times \mathbb{R}^{\dim M} \cong_{\text{set}} \mathbb{R}^{2\dim M} \\ X &\mapsto (x(\pi(X)), X^1, \ldots, X^{\dim M}). \end{aligned}$$

Assuming that TM is equipped with a suitable topology, for instance the initial topology (i.e. the coarsest topology on TM that makes π continuous), we claim that the pair $(\mathrm{preim}_\pi(U), \xi)$ is a chart on TM and

$$\mathscr{A}_{TM} := \{(\mathrm{preim}_\pi(U), \xi) \mid (U,x) \in \mathscr{A}_M\}$$

is a smooth atlas on TM. Note that, from its definition, it is clear that ξ is a bijection. We will not show that $(\mathrm{preim}_\pi(U), \xi)$ is a chart here, but we will show that $\mathscr{A}_{TM}$ is a smooth atlas. The tangent bundle of a smooth manifold M is therefore itself a smooth manifold of dimension $2\dim M$, and the projection $\pi \colon TM \to M$ is smooth with respect to this structure.

Similarly, one can construct the *cotangent bundle* T^*M to M by defining

$$T^*M := \coprod_{p \in M} T_p^*M$$

and going through the above again, using the dual basis $\{(\mathrm{d}x^a)_p\}$ instead of $\{\big(\frac{\partial}{\partial x^a}\big)_p\}$.

4.2.2 Vector fields

Now that we have defined the tangent bundle, we are ready to define vector fields.

Remark 4.13. An equivalent definition is that a vector field σ on M is a derivation on the algebra $\mathcal{C}^\infty(M)$, i.e. an $\mathbb{R}$-linear map

$$\sigma \colon \mathcal{C}^\infty(M) \xrightarrow{\sim} \mathcal{C}^\infty(M)$$

satisfying the Leibniz rule (with respect to pointwise multiplication on $\mathcal{C}^\infty(M)$)

$$\sigma(fg) = g\,\sigma(f) + f\,\sigma(g).$$

This definition is better suited for some purposes, and later on we will switch from one to the other without making any notational distinction between them.

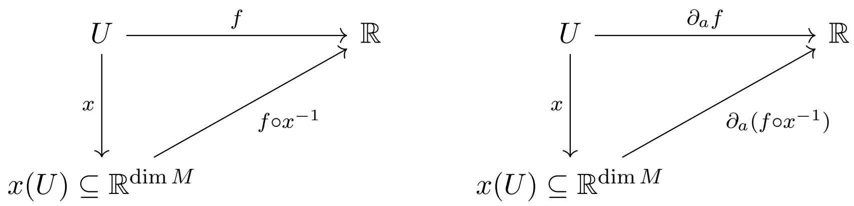
Example 4.14. Let (U, x) be a chart on M. For each $1 \leq a \leq \dim M$, the map

$$\begin{aligned} \sigma \colon U &\to TU \\ p &\mapsto \left(\frac{\partial}{\partial x^a}\right)_p \end{aligned}$$

is a vector field on the submanifold U. We can also think of this as a linear map

$$\begin{aligned} \frac{\partial}{\partial x^a} \colon \mathcal{C}^\infty(U) &\xrightarrow{\sim} \mathcal{C}^\infty(U) \\ f &\mapsto \frac{\partial}{\partial x^a}(f) = \partial_a(f \circ x^{-1}) \circ x. \end{aligned}$$

By abuse of notation, one usually denotes the right hand side above simply as $\partial_a f$.



Recall that, given a smooth map $\phi \colon M \to N$, the push-forward $(\phi_*)_p$ is a linear map that takes in a tangent vector in T_pM and outputs a tangent vector in $T_{\phi(p)}N$. Of course, we have one such map for each $p \in M$. We can collect them all into a single smooth map.

4.2.3 Differential forms

Another way to define a differential form, besides the one discussed in the appendix is as a smooth section of the appropriate bundle on M, i.e. as a map assigning to each $p \in M$ an n-form on the vector space T_pM.

Example 4.15. a) A manifold M is said to be *orientable* if it admits an oriented atlas, i.e. an atlas in which all chart transition maps, which are maps between open subsets of $\mathbb{R}^{\dim M}$, have a positive determinant.

If M is orientable, then there exists a nowhere vanishing top form ($n = \dim M$) on M providing the volume.

b) The electromagnetic field strength F is a differential 2-form built from the electric and magnetic fields, which are also taken to be forms. We will define these later in some detail.

c) In classical mechanics, if Q is a smooth manifold describing the possible system configurations, then the phase space is T^*Q. There exists a canonically defined 2-form on T^*Q known as a symplectic form, which we will define later.

If ω is an n-form, then n is said to be the *degree* of ω. We denote by $\Omega^n(M)$ the set of all differential n-forms on M, which then becomes a $\mathcal{C}^\infty(M)$-module by defining the addition and multiplication operations pointwise.

Example 4.16. Of course, we have $\Omega^0(M) \equiv \mathcal{C}^\infty(M)$ and $\Omega^1(M) \equiv \Gamma(T^0_1 M) \equiv \Gamma(T^*M)$.

Similarly to the case of forms on vector spaces, we have $\Omega^n(M) = \{0\}$ for $n > \dim M$, and otherwise $\dim \Omega^n(M) = \binom{\dim M}{n}$, as a $\mathcal{C}^\infty(M)$-module.

We can specialize the pull-back of tensors to differential forms. This works for any smooth map ϕ, and it leads to the mantra:

Vectors are pushed forward,
forms are pulled back.

The tensor product $\otimes$ does not interact well with forms, since the tensor product of two forms is not necessarily a form. Recall the definition of the gradient operator at a point $p \in M$. We can extend that definition to define the ($\mathbb{R}$-linear) operator:

$$\begin{aligned} \mathrm{d} \colon \mathcal{C}^\infty(M) &\xrightarrow{\sim} \Gamma(T^*M) \\ f &\mapsto \mathrm{d}f \end{aligned}$$

where, of course, $\mathrm{d}f \colon p \mapsto \mathrm{d}_p f$. Alternatively, we can think of $\mathrm{d}f$ as the $\mathbb{R}$-linear map

$$\begin{aligned} \mathrm{d}f \colon \Gamma(TM) &\xrightarrow{\sim} \mathcal{C}^\infty(M) \\ X &\mapsto \mathrm{d}f(X) = X(f). \end{aligned}$$

Remark 4.17. Locally on some chart (U, x) on M, the covector field (or 1-form) $\mathrm{d}f$ can be expressed as

$$\mathrm{d}f = \lambda_a \, \mathrm{d}x^a$$

for some smooth functions $\lambda_i \in \mathcal{C}^\infty(U)$. To determine what they are, we simply apply both sides to the vector fields induced by the chart. We have

$$\mathrm{d}f\left(\frac{\partial}{\partial x^b}\right) = \frac{\partial}{\partial x^b}(f) = \partial_b f$$

and

$$\lambda_a \, \mathrm{d}x^a\left(\frac{\partial}{\partial x^b}\right) = \lambda_a \, \frac{\partial}{\partial x^b}(x^a) = \lambda_a \, \delta^a_b = \lambda_b.$$

Hence, the local expression of $\mathrm{d}f$ on (U, x) is

$$\mathrm{d}f = \partial_a f \, \mathrm{d}x^a.$$

Note that the operator d satisfies the Leibniz rule

$$\mathrm{d}(fg) = g \, \mathrm{d}f + f \, \mathrm{d}g.$$

We can also understand this as an operator that takes in 0-forms and outputs 1-forms

$$\mathrm{d} \colon \Omega^0(M) \xrightarrow{\sim} \Omega^1(M).$$

This can then be extended to an operator which acts on any n-form.

Remark 4.18. Note that the operator d is only well-defined when it acts on forms. In order to define a derivative operator on general tensors we will need to add extra structure to our differentiable manifold.

Remark 4.19. Informally, we can write this result as $\Phi^*\mathrm{d} = \mathrm{d}\Phi^*$, and say that the exterior derivative "commutes" with the pull-back.

However, you should bear in mind that the two d's appearing in the statement are two different operators. On the left hand side, it is $\mathrm{d}\colon \Omega^n(N) \to \Omega^{n+1}(N)$, while it is $\mathrm{d}\colon \Omega^n(M) \to \Omega^{n+1}(M)$ on the right hand side.

Remark 4.20. Of course, we could also combine the operators d into a single operator acting on the Grassmann algebra on M

$$\mathrm{d}\colon \Omega(M) \to \Omega(M)$$

by linear continuation.

Example 4.21. In the modern formulation of Maxwell's electrodynamics, the electric and magnetic fields E and B are taken to be a 1-form and a 2-form on $\mathbb{R}^3$, respectively:

$$\begin{aligned} E &:= E_x \mathrm{d}x + E_y \mathrm{d}y + E_z \mathrm{d}z \\ B &:= B_x \mathrm{d}y \wedge \mathrm{d}z + B_y \mathrm{d}z \wedge \mathrm{d}x + B_z \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

The electromagnetic field strength F is then defined as the 2-form on $\mathbb{R}^4$

$$F := B + E \wedge \mathrm{d}t.$$

In components, we can write

$$F = F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu,$$

where $(\mathrm{d}x^0, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3) \equiv (\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ and

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

The field strength satisfies the equation

$$\mathrm{d}F = 0.$$

This is called the homogeneous Maxwell's equation and it is, in fact, equivalent to the two homogeneous Maxwell's (vectorial) equations

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= \mathbf{0}. \end{aligned}$$

In order to cast the remaining Maxwell's equations into the language of differential forms, we need a further operation on forms, called the Hodge star operator.

Recall from the standard theory of electrodynamics that the two equations above imply the existence of the electric and vector potentials φ and $\mathbf{A} = (A_x, A_y, A_z)$, satisfying

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}.$$

Similarly, the equation $\mathrm{d}F = 0$ on $\mathbb{R}^4$ implies the existence of an electromagnetic 4-potential (or gauge potential) form $A \in \Omega^1(\mathbb{R}^4)$ such that

$$F = \mathrm{d}A.$$

Indeed, we can take

$$A := -\varphi\,\mathrm{d}t + A_x\mathrm{d}x + A_y\mathrm{d}y + A_z\mathrm{d}z.$$

Example 4.22. In the Hamiltonian formulation of classical mechanics one is especially interested in the cotangent bundle T^*Q of some configuration space Q. Similarly to what we did when we introduced the tangent bundle, we can define (at least locally) a system of coordinates on T^*Q by

$$(q^1, \dots, q^{\dim Q}, p_1, \dots, p_{\dim Q}),$$

where the p_i's are the generalised momenta on Q and the q^i's are the generalised coordinates on Q (recall that $\dim T^*Q = 2\dim Q$). We can then define a 1-form $\theta \in \Omega^1(T^*Q)$ by

$$\theta := p_i\,\mathrm{d}q^i$$

called the symplectic potential. If we further define

$$\omega := \mathrm{d}\theta \in \Omega^2(T^*Q),$$

then we can calculate that

$$\mathrm{d}\omega = \mathrm{d}(\mathrm{d}\theta) = \cdots = 0.$$

Moreover, ω is non-degenerate and hence it is a symplectic form on T^*Q.

4.2.4 Lie group actions

Remark 4.23. From the definition given in the appendix, the smooth structures on G and M were only used in the requirement that $\triangleright$ be smooth. By dropping this condition, we obtain the usual definition of a group action on a set. Some of the definitions that we will soon give for Lie groups and smooth manifolds, such as those of orbits and stabilizers, also have clear analogues to the case of bare groups and sets.

Example 4.24. Let G be a Lie group and let $R\colon G \to \mathrm{GL}(V)$ be a representation of G on a vector space V. Define a map

$$\begin{aligned} \triangleright\colon G \times V &\to V \\ (g, v) &\mapsto g \triangleright v := R(g)v. \end{aligned}$$

We easily check that $e \triangleright v := R(e)v = \mathrm{id}_V\, v = v$ and

$$\begin{aligned}(g_1 \bullet g_2) \triangleright v &:= R(g_1 \bullet g_2)v \\ &= (R(g_1) \circ R(g_2))v \\ &= R(g_1)(R(g_2)v) \\ &= g_1 \triangleright (g_2 \triangleright v),\end{aligned}$$

for any $v \in V$ and any $g_1, g_2 \in G$. Moreover, if we equip V with the usual smooth structure, the map $\triangleright$ is smooth and hence a Lie group action on V. It follows that representations of Lie groups are just a special case of left Lie group actions. We can therefore think of left G-actions as generalised representations of G on some manifold.

Remark 4.25. Since for each $g \in G$ we also have $g^{-1} \in G$, if we need *some* action of G on M, then a left action is just as good as a right action. Only later, within the context of principal and associated fibre bundles, we will attach separate "meanings" to left and right actions. Some of the next definitions and results will only be given in terms of left actions, but they obviously apply to right actions as well.

Remark 4.26. Recall that if we have a basis $e_1, \ldots, e_{\dim M}$ of T_pM and $X^1, \ldots, X^{\dim M}$ are the components of some $X \in T_pM$ in this basis, then under a change of basis

$$\widetilde{e}_a = A^b{}_a e_b,$$

we have $X = \widetilde{X}^a \widetilde{e}_a$, where

$$\widetilde{X}^a = (A^{-1})^a{}_b X^b.$$

Once expressed in terms of principal and associated fibre bundles, we will see that the "recipe" of labelling the basis by lower indices and the vector components by upper indices, as well as their transformation law, can be understood as a right action of $\mathrm{GL}(\dim M, \mathbb{R})$ on the basis and a left action of the same $\mathrm{GL}(\dim M, \mathbb{R})$ on the components.

4.2.5 Principal fiber bundles

We can specialize our definition of bundle to define a *smooth bundle*, which is just a bundle (E, π, M) where E and M are smooth manifolds and the projection $\pi \colon E \to M$ is smooth. Two smooth bundles (E, π, M) and (E', π', M') are isomorphic if there exist diffeomorphisms u, f such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\;u\;} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{\;f\;} & M' \end{array}$$

It is said that, roughly speaking, a principal bundle is a bundle whose fibre at each point is a Lie group. Note that the formal definition is that a principal G-bundle is a bundle which is isomorphic to a bundle whose fibres are the orbits under the right action of G, which are themselves isomorphic to G since the action is free.

The frame bundle will be a prototypical example of a principal bundle.

Example 4.27. a) Let M be a smooth manifold. Consider the space

$$L_pM := \{(e_1, \dots, e_{\dim M}) \mid e_1, \dots, e_{\dim M} \text{ is a basis of } T_pM\} \cong_{\text{vec}} \mathrm{GL}(\dim M, \mathbb{R}).$$

We know from linear algebra that the bases of a vector space are related to each other by invertible linear transformations. Hence, we have

$$L_pM \cong_{\text{vec}} \mathrm{GL}(\dim M, \mathbb{R}).$$

We define the frame bundle of M as

$$LM := \coprod_{p \in M} L_pM$$

with the obvious projection map $\pi \colon LM \to M$ sending each basis $(e_1, \dots, e_{\dim M})$ to the unique point $p \in M$ such that $(e_1, \dots, e_{\dim M})$ is a basis of T_pM. By proceeding similarly to the case of the tangent bundle, we can equip LM with a smooth structure inherited from that of M. We then find

$$\dim LM = \dim M + \dim T_pM = \dim M + (\dim M)^2.$$

b) We would now like to make $LM \xrightarrow{\pi} M$ into a principal $\mathrm{GL}(\dim M, \mathbb{R})$-bundle. We define a right $\mathrm{GL}(\dim M, \mathbb{R})$-action on LM by

$$(e_1, \dots, e_{\dim M}) \lhd g := (g^a{}_1 e_a, \dots, g^a{}_{\dim M} e_a),$$

where $g^a{}_b$ are the components of the endomorphism $g \in \mathrm{GL}(\dim M, \mathbb{R})$ with respect to the standard basis on $\mathbb{R}^n$. Note that if $(e_1, \dots, e_{\dim M}) \in L_pM$, we must also have $(e_1, \dots, e_{\dim M}) \lhd g \in L_pM$. This action is free since

$$(e_1, \dots, e_{\dim M}) \lhd g = (e_1, \dots, e_{\dim M}) \Leftrightarrow (g^a{}_1 e_a, \dots, g^a{}_{\dim M} e_a) = (e_1, \dots, e_{\dim M})$$

and hence, by linear independence, $g^a{}_b = \delta^a_b$, so $g = \mathrm{id}_{\mathbb{R}^n}$. Note that since all bases of each T_pM are related by some $g \in \mathrm{GL}(\dim M, \mathbb{R})$, $\lhd$ is also fibre-wise transitive.

c) We now have to show that

$$\begin{array}{c} LM \\ \pi \downarrow \\ M \end{array} \quad \cong_{\text{bdl}} \quad \begin{array}{c} LM \\ \downarrow \rho \\ LM/\,\mathrm{GL}(\dim M, \mathbb{R}) \end{array}$$

i.e. that there exist smooth maps u and f such that the diagram

$$\begin{array}{ccc} LM & \overset{u}{\underset{u^{-1}}{\rightleftarrows}} & LM \\ \pi \downarrow & & \downarrow \rho \\ M & \overset{f}{\underset{f^{-1}}{\rightleftarrows}} & LM/\,\mathrm{GL}(\dim M, \mathbb{R}) \end{array}$$

commutes. We can simply choose $u = u^{-1} = \mathrm{id}_{LM}$, while we define f as

$$\begin{aligned} f\colon M &\to LM/\,\mathrm{GL}(\dim M, \mathbb{R}) \\ p &\mapsto \mathrm{GL}(\dim M, \mathbb{R})_{(e_1,\dots,e_{\dim M})}, \end{aligned}$$

where $(e_1, \dots, e_{\dim M})$ is some basis of T_pM, i.e. $(e_1, \dots, e_{\dim M}) \in \mathrm{preim}_\pi(\{p\})$. Note that f is well-defined since every basis of T_pM gives rise to the same orbit in the orbit space $LM/\,\mathrm{GL}(\dim M, \mathbb{R})$. Moreover, it is injective since

$$f(p) = f(p') \;\Leftrightarrow\; \mathrm{GL}(\dim M, \mathbb{R})_{(e_1,\dots,e_{\dim M})} = \mathrm{GL}(\dim M, \mathbb{R})_{(e'_1,\dots,e'_{\dim M})},$$

which is true only if $(e_1, \dots, e_{\dim M})$ and $(e'_1, \dots, e'_{\dim M})$ are basis of the same tangent space, so $p = p'$. It is clearly surjective since every orbit in $LM/\,\mathrm{GL}(\dim M, \mathbb{R})$ is the orbit of some basis of some tangent space T_pM at some point $p \in M$. The inverse map is given explicitly by

$$\begin{aligned} f^{-1}\colon \quad LM/\,\mathrm{GL}(\dim M, \mathbb{R}) &\to M \\ \mathrm{GL}(\dim M, \mathbb{R})_{(e_1,\dots,e_{\dim M})} &\mapsto \pi((e_1, \dots, e_{\dim M})). \end{aligned}$$

Finally, we have

$$(\rho \circ \mathrm{id}_{LM})(e_1, \dots, e_{\dim M}) = \mathrm{GL}(\dim M, \mathbb{R})_{(e_1,\dots,e_{\dim M})} = (f \circ \pi)(e_1, \dots, e_{\dim M})$$

and thus $LM \xrightarrow{\pi} M$ is a principal G-bundle, called the *frame bundle* of M.

Example 4.28. The existence of a section on the frame bundle LM can be reduced to the existence of $(\dim M)$ non-everywhere vanishing linearly independent vector fields on M. Since no such vector field exists on even-dimensional spheres, LS^{2n} is always non-trivial.

4.2.6 Associated fiber bundle

An associated fibre bundle is a fibre bundle which is associated (in a precise sense) to a principal G-bundle. Associated bundles are related to their underlying principal bundles in a way that models the transformation law for components under a change of basis.

The *associated bundle* (to (P, π, M), F and $\triangleright$) is the bundle (P_F, π_F, M).

Example 4.29. Recall that the frame bundle (LM, π, M) is a principal $\mathrm{GL}(d, \mathbb{R})$-bundle, where $d = \dim M$, with right G-action $\triangleleft\colon LM \times G \to LM$ given by

$$(e_1, \dots, e_d) \triangleleft g := (g^a{}_1 e_a, \dots, g^a{}_d e_a).$$

Let $F := \mathbb{R}^d$ (as a smooth manifold) and define a left action

$$\begin{aligned} \triangleright\colon \mathrm{GL}(d, \mathbb{R}) \times \mathbb{R}^d &\to \mathbb{R}^d \\ (g, x) &\mapsto g \triangleright x, \end{aligned}$$

where

$$(g \triangleright x)^a := g^a{}_b x^b.$$

Then $(LM_{\mathbb{R}^d}, \pi_{\mathbb{R}^d}, \mathbb{R}^d)$ is the associated bundle. In fact, we have a bundle isomorphism

$$\begin{array}{ccc} LM_{\mathbb{R}^d} & \xrightarrow{\quad u \quad} & TM \\ {\scriptstyle \pi_{\mathbb{R}^d}}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\quad \mathrm{id}_M \quad} & M \end{array}$$

where (TM, π, M) is the tangent bundle of M, and u is defined as

$$\begin{aligned} u\colon \qquad LM_{\mathbb{R}^d} \qquad &\to TM \\ [(e_1, \ldots, e_d), x] &\mapsto x^a e_a. \end{aligned}$$

The inverse map $u^{-1}\colon TM \to LM_{\mathbb{R}^d}$ works as follows. Given any $X \in TM$, pick any basis $(e_1, \ldots, e_d)$ of the tangent space at the point $\pi(X) \in M$, i.e. any element of $L_{\pi(X)}M$. Decompose X as $x^a e_a$, with each $x^a \in \mathbb{R}$, and define

$$u^{-1}(X) := [(e_1, \ldots, e_d), x].$$

The map u^{-1} is well-defined since, while the pair $((e_1, \ldots, e_d), x) \in LM \times \mathbb{R}^d$ clearly depends on the choice of basis, the equivalence class

$$[(e_1, \ldots, e_d), x] \in LM_{\mathbb{R}^d} := (LM \times \mathbb{R}^d)/{\sim_G}$$

does not. It includes all pairs $((e_1, \ldots, e_d) \triangleleft g, g^{-1} \triangleright x)$ for every $g \in \mathrm{GL}(d, \mathbb{R})$, i.e. every choice of basis together with the "right" components $x \in \mathbb{R}^d$.

Remark 4.30. Even though the associated bundle $(LM_{\mathbb{R}^d}, \pi_{\mathbb{R}^d}, \mathbb{R}^d)$ is isomorphic to the tangent bundle (TM, π, M), note a subtle difference between the two. On the tangent bundle, the transformation law for a change of basis and the related transformation law for components are *deduced* from the definitions by undergraduate linear algebra.

On the other hand, the transformation laws on $LM_{\mathbb{R}^d}$ were *chosen* by us in its definition. We chose the Lie group $\mathrm{GL}(d, \mathbb{R})$, the specific right action $\triangleleft$ on LM, the space $\mathbb{R}^d$, and the specific left action on $\mathbb{R}^d$. It just happens that, with these choices, the resulting associated bundle is isomorphic to the tangent bundle. Of course, we have the freedom to make different choices and construct bundles which behave very differently from TM.

Example 4.31. Consider the principal $\mathrm{GL}(d, \mathbb{R})$-bundle (LM, π, M) again, with the same right action as before. This time we define

$$F := (\mathbb{R}^d)^{\times p} \times (\mathbb{R}^{d^*})^{\times q} := \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{p \text{ times}} \times \underbrace{\mathbb{R}^{d^*} \times \cdots \times \mathbb{R}^{d^*}}_{q \text{ times}}$$

with left $\mathrm{GL}(d, \mathbb{R})$-action $\triangleright\colon \mathrm{GL}(d, \mathbb{R}) \times F \to F$ given by

$$(g \triangleright f)^{a_1 \cdots a_p}{}_{b_1 \cdots b_q} := g^{a_1}{}_{\widetilde{a}_1} \cdots g^{a_p}{}_{\widetilde{a}_p} (g^{-1})^{\widetilde{b}_1}{}_{b_1} \cdots (g^{-1})^{\widetilde{b}_q}{}_{b_q} f^{\widetilde{a}_1 \cdots \widetilde{a}_p}{}_{\widetilde{b}_1 \cdots \widetilde{b}_q}.$$

Then, the associated bundle (LM_F, π_F, M) thus constructed is isomorphic to $(T^p_q M, \pi, M)$, the (p, q)-tensor bundle on M.

Remark 4.32. Some special cases include the following.

i) If $\omega = 0$, we recover the (p, q)-tensor bundle on M.

ii) If $F = \mathbb{R}$ (i.e. $p = q = 0$), the left action reduces to

$$(g \triangleright f) = (\det g^{-1})^{\omega} f,$$

which is the transformation law for a *scalar density of weight* ω.

iii) If $\mathrm{GL}(d, \mathbb{R})$ is restricted in such a way that we always have $(\det g^{-1}) = 1$, then tensor densities are indistinguishable from ordinary tensor fields. This is why you probably haven't met tensor densities in your special relativity course.

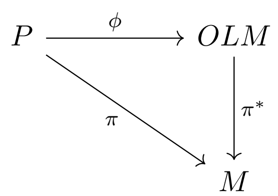
Example 4.33. Recall that if B is a bilinear form on a K-vector space V, the determinant of B is not independent from the choice of basis. Indeed, if $\{e_a\}$ and $\{e'_b := g^a{}_b e_a\}$ are both basis of V, where $g \in \mathrm{GL}(\dim V, K)$, then

$$(\det B)' = (\det g^{-1})^2 \det B.$$

Once recast in the principal and associated bundle formalism, we find that the determinant of a bilinear form is a scalar density of weight 2.

4.2.7 Application

Spinors are hard to define, paraphrasing a joke of sir. M. Atiyah: "...only God and maybe Dirac know what they are...". The principal and associated bundle formalism gives a geometric interpretation and a way to imagine what a spinor is: namely a spinor is an element of the spinor field, the spinor field is a sections of an appropriate bundle called the spin-bundle. The spin-bundle is associated to a suitable bundle called the principal spin frame bundle. This is in complete analogy with a matter field as section of the tangent bundle that is associated to the principal frame bundle. To achieve that, we need first to remember the spin groups as double cover of the special orthogonal groups and construct a Lie group homomorphism. The definition of a spin frame bundle over an n-dimensional manifold requires the introduction of a metric, namely Riemannian metric, in order to define the spin structure (P,ϕ). ϕ which is a map from the total space P to the orthogonal frame bundle OLM, such that the following digram commutes:



and the bundle morphism ϕ is supposed to be ρ-equivariant as described before. A remark has to be made for the existence of the spin structure.

Theorem 4.34. *A Riemannian (M,g) admits a spin structure if and only if the second Stiefel-Whitney class vanishes.*

This is a topological invariant, you can always decide whether you can put such a spin structure on a Riemannian manifold. For special cases:

Theorem 4.35. *If the dimension of M is less or equal to 3 and the manifold is both compact and orientable, then a spin structure always exists.*

Example 4.36. In quantum mechanics on a 2-sphere, the spin structure always exists.

Definition. A spinor field is a section of a spin bundle.

Which is not the just defined spin frame bundle.

Definition. A spin bundle is a vector bundle, with typical fiber F associated to the spin frame bundle by virtue of a linear left action that is a linear representation of spin.

Therefor F is a representation of Spin() and the spin bundle is the associated bundle to the principal spin frame bundle. In practice, what we mean is that doing something in the spin frame (P), it has a bearing on the orthogonal frame (OLM). This is what happens in physics all the time, we rotate the measurement apparatus macroscopically (which is not spin) say by a 2π rotation in the OLM, however in P we just went from the identity to half the full angle.

Remark 4.37. In principle we need not to restrict ourselves in the orthogonal bundle, however it is exactly that bundle that is kept invariant under the action of the special orthogonal groups.

What if we didn't take the orthogonal frame bundle? That is we do not ask for a metric, then we have the total frame bundle which is a principal general linear bundle and the question is: are there double covering maps? In other words, are there spins corresponding to the general linear transformations, like there are spins corresponding to the special orthogonal groups? And the answer is no, there aren't. Our notion of spinors comes from an underlying metric, in mathematics tensors are more elementary because they need just a smooth manifold. On the other end, going from tensor to spin, we already need a metric geometry underlying in order to even define spin. In physics the opposite is assumed, because one can construct tensors from spinors, but the misunderstanding is now clear: we don't really understand the full frame bundle, but once we restrict the attention to orthogonal frame bundles we can understand spinors. Therefore if and only if you are given a Riemannian manifold, that has already a definition of metric, then a spin is more elementary than a tensor. We have a frame bundle and sections of an associated line bundle, to define the spin covariant derivative, . Having sections of an associated line bundle and wanting to construct the spin covariant derivative, as found in general relativity books we need the concept of equivalence of local sections and equivariant functions on the principal frame bundle that will be defined in the next sections. In words, the sections of the associated bundle can be made in a spin(n)-equivariant fiber valued functions of the spin frame bundle, then push everything down on the manifold by choosing a particular section on the spin bundle and you get the local version in terms of Yang-Mills fields. In GR the formula for the spin covariant derivative, is the same construction for the general covariant derivatives of sections of associated bundles. The problem reduces to choose a connection on the principal spin(n) bundle.

4.3 Connections

In elementary courses on differential geometry or general relativity, the notions of connection, parallel transport and covariant derivative are often confused with one another. Sometimes, the terms are even used as synonyms, but this is not the way to understand the different terms. What a connection really is, can be interpreted as just additional structure on a principal bundle consisting in a "smooth" assignment of a particular vector space at each point of the base manifold compatible with the right action of the Lie group on the principal bundle. Such an assignment is, in fact, equivalent to a certain Lie-algebra-valued one-form on the principal bundle, as we will discuss below. Later, we will see that a connection on a principal bundle induces a parallel transport map on the principal bundle, which in turn induces a parallel transport map on any of its associated bundles. If the fibres of the associated bundle carry a vector space structure, then the parallel transport can be used to define a covariant derivative on the associated bundle.

Hence the conceptual sequence "connection, parallel transport covariant derivative" is in decreasing order of generality, and it should be clear that treating these terms as synonyms will inevitably lead to confusion.

Let (P, π, M) be a principal G-bundle. Recall that every element of $A \in T_eG$ gives rise to a left invariant vector field on G which we denoted by X^A. However, we will now reserve this notation for a vector field on P instead. Given $A \in T_eG$, we define $X^A \in \Gamma(TP)$ by

$$\begin{aligned} X^A_p \colon \mathcal{C}^\infty(P) &\xrightarrow{\sim} \mathbb{R} \\ f &\mapsto [f(p \lhd \exp(tA))]'(0), \end{aligned}$$

where the derivative is to be taken with respect to t. We also define the maps

$$\begin{aligned} i_p \colon T_eG &\to T_pP \\ A &\mapsto X^A_p, \end{aligned}$$

which can be shown to be a Lie algebra homomorphism.

The idea of a connection is to make a choice of how to "connect" the individual points in "neighbouring" fibres in a principal fibre bundle. The definition formalises the idea that the assignment of an H_pP to each $p \in P$ should be "smooth" within each fibre (i) as well as between different fibres (ii).

Remark 4.38. For each $X_p \in T_pP$, both $\mathrm{hor}(X_p)$ and $\mathrm{ver}(X_p)$ depend on the choice of H_pP.

4.3.1 Connection one-forms

Technically, the choice of a horizontal subspace H_pP at each $p \in P$ providing a connection is conveniently encoded in the thus induced Lie-algebra-valued one-form

$$\begin{aligned} \omega_p \colon T_pP &\xrightarrow{\sim} T_eG \\ X_p &\mapsto \omega_p(X_p) := i_p^{-1}(\mathrm{ver}(X_p)) \end{aligned}$$

Remark 4.39. We have seen how to produce a one-form from a choice of horizontal spaces (i.e. a connection). The choice of horizontal spaces can be recovered from ω by

$$H_pP = \ker(\omega_p).$$

Of course, not every (Lie-algebra-valued) one-form on P is such that $\ker(\omega_p)$ gives a connection on the principal bundle. What we would now like to do is to study some crucial properties of ω. We will then elevate these properties to a definition of connection one-form absent a connection, so that we may re-define the notion of connection in terms of a connection one-form.

4.3.2 Local representation of a connection one-forms

After understanding how to associate a connection one-form to a connection, i.e. a certain Lie-algebra-valued one-form to a smooth choice of horizontal spaces on the principal bundle. We will now study how we can express this connection one-form locally on the base manifold of the principal bundle.

Recall that a connection one-form on a principal bundle (P, π, M) is a smooth Lie-algebra-valued one-form, i.e. a smooth map

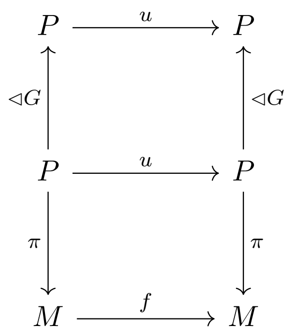
$$\omega\colon \Gamma(TP) \xrightarrow{\sim} T_eG$$

which "behaves like a one-form", in the sense that it is $\mathbb{R}$-linear and satisfies the Leibniz rule, and such that, in addition, for all $A \in T_eG$, $g \in G$ and $X \in \Gamma(TP)$, we have

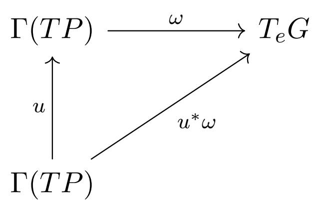
i) $\omega(X^A) = A$;

ii) $((\lhd\, g)^*\omega)(X) = (\mathrm{Ad}_{g^{-1}})_*(\omega(X))$.

If the pair (u, f) is a principal bundle automorphism of (P, π, M), i.e. if the diagram



commutes, we should be able to pull a connection one-form ω on P back to another connection one-form $u^*\omega$ on P.



Recall that for a one-form $\omega\colon \Gamma(TN) \xrightarrow{\sim} \mathcal{C}^\infty(N)$, we defined

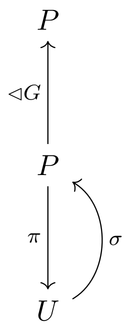
$$\begin{aligned}\Phi^*(\omega)\colon \Gamma(TM) &\xrightarrow{\sim} \mathcal{C}^\infty(M)\\ X &\mapsto \omega(\Phi_*(X)) \circ \phi\end{aligned}$$

for any diffeomorphism $\phi\colon M \to N$. One might be worried about whether this and similar definitions apply to Lie-algebra-valued one-forms but, in fact, they do. In our case, even though ω lands in T_eG, its domain is still $\Gamma(TP)$ and if $u\colon P \to P$ is a diffeomorphism of P, then $u_*X \in \Gamma(TP)$ and so

$$u^*\omega\colon X \mapsto (u^*\omega)(X) := \omega(u_*(X)) \circ u$$

is again a Lie-algebra-valued one-form. Note that we will no longer distinguish notationally between the push-forward of tangent vectors and of vector fields.

In practice, e.g. for calculational purposes, one may wish to restrict attention to some open subset $U \subseteq M$. Let $\sigma\colon U \to P$ be a local section of P, i.e. $\pi \circ \sigma = \mathrm{id}_U$.



The next definition is so important that is copy and pasted directly from the appendix:

Definition. Given a connection one-form ω on P, such a local section σ induces

i) a *Yang-Mills field* $\omega^U : \Gamma(TU) \xrightarrow{\sim} T_eG$ given by

$$\omega^U := \sigma^*\omega;$$

ii) a *local trivialisation* of the principal bundle P, i.e. a map

$$\begin{aligned} h\colon U \times G &\to P \\ (m, g) &\mapsto \sigma(m) \lhd g; \end{aligned}$$

iii) a *local representation* of ω on U by

$$h^*\omega\colon \Gamma(T(U \times G)) \xrightarrow{\sim} T_eG.$$

Note that, at each point $(m, g) \in U \times G$, we have

$$T_{(m,g)}(U \times G) \cong_{\text{Lie alg}} T_mU \oplus T_gG.$$

Remark 4.40. Both the Yang-Mills field ω^U and the local representation $h^*\omega$ encode the information carried by ω locally on U. Since $h^*\omega$ involves $U \times G$ while ω^U doesn't, one might guess that $h^*\omega$ gives a more "accurate" picture of ω on U than the Yang-Mills field. But in fact, this is not the case. They both contain the same amount of local information about the connection one-form ω.

4.3.3 Maurer-Cartan form

Th relation between the Yang-Mills field and the local representation is provided by the introduction of the Ξ_g, which is the Maurer-Cartan form.

Remark 4.41. Note that we have represented a generic element of T_gG as L_g^A. This is due to the following. Recall that the left translation map $\ell_g\colon G \to G$ is a diffeomorphism of G. As such, its push-forward at any point is a linear isomorphism. In particular, we have

$$((\ell_g)_*)_e\colon T_eG \xrightarrow{\sim} T_gG,$$

that is, the tangent space at any point $g \in G$ can be canonically identified with the tangent space at the identity. Hence, we can write any element of T_gG as

$$L_g^A := ((\ell_g)_*)_e(A)$$

for some $A \in T_eG$.

Let us consider some specific examples.

Example 4.42. Any chart (U, x) of a smooth manifold M induces a local section $\sigma\colon U \to LM$ of the frame bundle of M by

$$\sigma(m) := \left(\left(\frac{\partial}{\partial x^1}\right)_m, \dots, \left(\frac{\partial}{\partial x^{\dim M}}\right)_m\right) \in L_mM.$$

Since $\mathrm{GL}(\dim M, \mathbb{R})$ can be identified with an open subset of $\mathbb{R}^{(\dim M)^2}$, we have

$$T_e\,\mathrm{GL}(\dim M, \mathbb{R}) \cong_{\text{Lie alg}} \mathbb{R}^{(\dim M)^2},$$

where $\mathbb{R}^{(\dim M)^2}$ is understood as the algebra of $\dim M \times \dim M$ square matrices, with bracket induced by matrix multiplication. In fact, this holds for any open subset of a vector space, when considered as a smooth manifold. A connection one-form

$$\omega\colon \Gamma(LM) \xrightarrow{\sim} T_e\,\mathrm{GL}(\dim M, \mathbb{R})$$

can thus be given in terms of $(\dim M)^2$ functions

$$\omega^i{}_j\colon \Gamma(LM) \xrightarrow{\sim} \mathbb{R}, \qquad 1 \leq i, j \leq \dim M.$$

The associated Yang-Mills field $\omega^U := \sigma^*\omega$ is, at each point $m \in U$, a Lie-algebra-valued one-form on the vector space T_mU. By using the co-ordinate induced basis and its dual basis, we can express $(\omega^U)_m$ in terms of components as

$$(\omega^U)_m = \omega^U_\mu(m)\,(\mathrm{d}x^\mu)_m,$$

where $1 \leq \mu \leq \dim M$ and

$$\omega^U_\mu(m) := (\omega^U)_m\left(\left(\frac{\partial}{\partial x^\mu}\right)_m\right).$$

Since $(\omega^U)_m \colon T_m U \xrightarrow{\sim} T_e G$, we have $\omega^U_\mu \colon U \to T_e G$. Hence, by employing he same isomorphism as above, we can identify each $\omega^U_\mu(m)$ with a square $\dim M \times \dim M$ matrix and define the symbol

$$\Gamma^i{}_{j\mu}(m) := (\omega^U(m))^i{}_{j\mu} := (\omega^U_\mu(m))^i{}_j,$$

usually referred to as the *Christoffel symbol*. The middle term is just an alternative notation for the right-most side. Note that, even though all three indices i, j, μ run from 1 to $\dim M$, the numbers $\Gamma^i{}_{j\mu}(m)$ do not constitute the components of a $(1,2)$-tensor on U. Only the μ index transforms as a one-form component index, i.e.

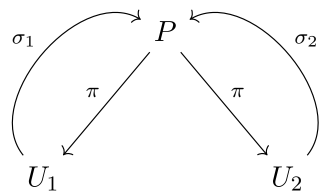
$$((g \triangleright \omega^U(m))^i{}_j)_\mu = (g^{-1})^\nu (\omega^U(m))^i{}_{j\nu}$$

for $g \in \mathrm{GL}(\dim M, \mathbb{R})$, while the i, j indices simply label different one-forms, $(\dim M)^2$ in total.

Note that the Maurer-Cartan form appearing in Theorem .107 only depends on the Lie group (and its Lie algebra), not on the principal bundle P or the restriction $U \subseteq M$. In the following example, we will go through the explicit calculation of the Maurer-Cartan form of the Lie group $\mathrm{GL}(d, \mathbb{R})$.

4.3.4 The gauge map

In physics, we are often prompted to write down a Yang-Mills field because we have local information about a connection. We can then try to reconstruct the global connection by glueing the Yang-Mills fields on several open subsets of our manifold.



Suppose, for instance, that we have two open subsets $U_1, U_2 \subseteq M$ and consider the Yang-Mills fields associated to two local connections σ_1, σ_2. If ω^{U_1} and ω^{U_2} are both local versions of a unique connection one-form, then is $U_1 \cap U_2 \neq \varnothing$, the Yang-Mills fields ω^{U_1} and ω^{U_2} should satisfy some compatibility condition on $U_1 \cap U_2$.

Example 4.43. Consider again the frame bundle LM of some manifold M. Let us evaluate explicitly the pull-back along Ω of the Maurer-Cartan form. Since $\Xi_g \colon T_g G \to T_e G$ and $\Omega \colon U_1 \cap U_2 \to T_e G$, we have $\Omega^* \Xi_g \colon T(U_1 \cap U_2) \to T_e G$. Let x be a chart map near the point

$m \in U_1 \cap U_2$. We have

$$\begin{aligned}
((\Omega^*\Xi_g)_m)^i{}_j\left(\left(\frac{\partial}{\partial x^\mu}\right)_m\right) &= (\Xi_{\Omega(m)})^i{}_j\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_m\right)\\
&= (\Omega(m)^{-1})^i{}_k(\mathrm{d}\widetilde{x}^k{}_j)_{\Omega(m)}\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_m\right)\\
&= (\Omega(m)^{-1})^i{}_k\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_m\right)(\widetilde{x}^k{}_j)\\
&= (\Omega(m)^{-1})^i{}_k\left(\frac{\partial}{\partial x^\mu}\right)_m(\widetilde{x}^k{}_j \circ \Omega)\\
&= (\Omega(m)^{-1})^i{}_k\left(\frac{\partial}{\partial x^\mu}\right)_m(\Omega(m))^k{}_j.
\end{aligned}$$

hence, we can write

$$\begin{aligned}
((\Omega^*\Xi_g)_m)^i{}_j &= (\Omega(m)^{-1})^i{}_k\left(\frac{\partial}{\partial x^\mu}\right)_m(\Omega(m))^k{}_j\mathrm{d}x^\mu\\
&=: (\boldsymbol{\Omega}^{-1}\mathrm{d}\boldsymbol{\Omega})^i{}_j.
\end{aligned}$$

Let us now compute the other summand. Recall that Ad_g is the map

$$\begin{aligned}
\mathrm{Ad}_g \colon G &\to G\\
h &\mapsto g \bullet h \bullet g^{-1}
\end{aligned}$$

and since $\mathrm{Ad}_g(e) = e$, the push-forward $((\mathrm{Ad}_g)_*)_e \colon T_eG \xrightarrow{\sim} T_eG$ is a linear endomorphism of T_eG. Moreover, since here $G = \mathrm{GL}(d, \mathbb{R})$ is a matrix Lie group, we have

$$((\mathrm{Ad}_g)_*A)^i{}_j = g^i{}_kA^k{}_l(g^{-1})^l{}_j =: (\boldsymbol{g}\boldsymbol{A}\boldsymbol{g}^{-1})^i{}_j.$$

Hence, we have

$$(\mathrm{Ad}_{\Omega^{-1}(m)})_*(\omega^{U_1}) = (\Omega(m)^{-1})^i{}_k((\omega^{U_1})_m)^k{}_l(\Omega(m))^l{}_j$$

Altogether, we find that the transition rule for the Yang-Mills fields on the intersection of U_1 and U_2 is given by

$$(\omega^{U_2})^i{}_{j\mu} = (\Omega^{-1})^i{}_k(\omega^{U_1})^k{}_{l\mu}\Omega^l{}_j + (\Omega^{-1})^i{}_k\partial_\mu(\Omega^{-1})^k{}_j.$$

As an application, consider the spacial case in which the sections σ_1, σ_2 are induced by co-ordinate charts (U_1, x) and (U_2, y). Then we have

$$\begin{aligned}
\Omega^i{}_j &= \frac{\partial y^i}{\partial x^j} := \partial_j(y^i \circ x^{-1}) \circ x\\
(\Omega^{-1})^i{}_j &= \frac{\partial x^i}{\partial y^j} := \partial_j(x^i \circ y^{-1}) \circ y
\end{aligned}$$

and hence

$$(\omega^{U_2})^i{}_{j\nu} = \frac{\partial y^\mu}{\partial x^\nu}\left(\frac{\partial x^i}{\partial y^k}(\omega^{U_1})^k{}_{l\mu}\frac{\partial y^l}{\partial x^j} + \frac{\partial x^i}{\partial y^k}\frac{\partial^2 y^k}{\partial x^\mu \partial x^j}\right).$$

You may recognise this as the transformation law for the Christoffel symbols from general relativity.

4.4 Parallel transport

We now come to the second term in the sequence "connection, parallel transport, covariant derivative". The idea of parallel transport on a principal bundle hinges on that of horizontal lift of a curve on the base manifold, which is a lifting to a curve on the principal bundle in the sense that the projection to the base manifold of this curve gives the curve we started with. In particular, if the principal bundle is equipped with a connection, we would like to impose some extra conditions on this lifting, so that it "connects" nearby fibres in a nice way. We will then consider the same idea on an associated bundle and see how we can induce a derivative operator if the associated bundle is a vector bundle.

4.4.1 The horizontal lift

Intuitively, a horizontal lift of a curve γ on M is a curve $\gamma^\uparrow$ on P such that each point $\gamma^\uparrow(\lambda) \in P$ belongs to the fibre of $\gamma(\lambda)$ (condition i), the tangent vectors to the curve $\gamma^\uparrow$ have no vertical component (condition ii), i.e. they lie entirely in the horizontal spaces at each point, and finally the projection of the tangent vector to $\gamma^\uparrow$ at $\gamma^\uparrow(\lambda)$ coincides with the tangent vector to the curve γ at $\pi(\gamma^\uparrow(\lambda)) = \gamma(\lambda)$.

Remark 4.44. Note that the uniqueness in the above definition only stems from the choice of $p_0 \in \mathrm{preim}_\pi(\{\gamma(0)\})$. A curve on M has several horizontal lifts to a curve on P, but there is only one such curve going through each point $p_0 \in \mathrm{preim}_\pi(\{\gamma(0)\})$. Clearly, different horizontal lifts cannot intersect each other.

Our strategy to write down an explicit expression for the horizontal lift through $p_0 \in P$ of a curve $\gamma\colon [0,1] \to M$ is to proceed in two steps:

i) "Generate" the horizontal lift by starting from some arbitrary curve $\delta\colon [0,1] \to P$ such that $\pi \circ \delta = \gamma$ by action of a suitable curve $g\colon (0,1) \to G$ so that

$$\gamma^\uparrow(\lambda) = \delta(\lambda) \lhd g(\lambda).$$

The suitable curve g will be the solution to an ordinary differential equation with initial condition $g(0) = g_0$, where g_0 is the unique element in G such that

$$\delta(0) \lhd g_0 = p_0 \in P.$$

ii) We will explicitly solve (locally) this differential equation for $g\colon [0,1] \to P$ by a path-ordered integral over the local Yang-Mills field.

In order to further massage this ODE,

$$\dot g(\lambda) = -(\omega_{\delta(\lambda)}(X_{\delta,\delta(\lambda)}))g(\lambda).$$

let us consider a chart (U, x) on the base manifold M, such that the image of γ is entirely contained in U. A local section $\sigma\colon U \to P$ induces

i) a Yang-Mills field ω^U;

ii) a curve on P by $\delta := \sigma \circ \gamma$.

In fact, since the only condition imposed on δ is that $\pi \circ \delta = \gamma$, choosing a such a curve δ is equivalent to choosing a local section σ. Note that we have

$$\sigma_*(X_{\gamma,\gamma(\lambda)}) = X_{\delta,\delta(\lambda)},$$

and hence

$$\begin{aligned}
\omega_{\delta(\lambda)}(X_{\delta,\delta(\lambda)}) &= \omega_{\delta(\lambda)}(\sigma_*(X_{\gamma,\gamma(\lambda)})) \\
&= (\sigma^*\omega)_{\gamma(\lambda)}(X_{\gamma,\gamma(\lambda)}) \\
&= (\omega^U)_{\gamma(\lambda)}(X_{\gamma,\gamma(\lambda)}) \\
&= \omega^U_\mu(\gamma(\lambda))(\mathrm{d}x^\mu)_{\gamma(\lambda)}\left(X^\nu_\gamma(\gamma(\lambda))\left(\frac{\partial}{\partial x^\nu}\right)_{\gamma(\lambda)}\right) \\
&= \omega^U_\mu(\gamma(\lambda))X^\nu_\gamma(\gamma(\lambda))(\mathrm{d}x^\mu)_{\gamma(\lambda)}\left(\left(\frac{\partial}{\partial x^\nu}\right)_{\gamma(\lambda)}\right) \\
&= \omega^U_\mu(\gamma(\lambda))X^\nu_\gamma(\gamma(\lambda))\,\delta^\mu_\nu \\
&= \omega^U_\mu(\gamma(\lambda))X^\mu_\gamma(\gamma(\lambda)).
\end{aligned}$$

Thus, in the special case of a matrix Lie group, the ODE reads

$$\dot{g}(\lambda) = -\Gamma_\mu(\gamma(\lambda))\,\dot{\gamma}^\mu(\lambda),$$

where $\Gamma_\mu := \omega^U_\mu$ and $\dot{\gamma}^\mu(\lambda) := X^\mu_\gamma(\gamma(\lambda))$, together with the initial condition $g(0) = g_0$.

Remark 4.45. The parallel transport is, in fact, a bijection between the fibres $\mathrm{preim}_\pi(\{\gamma(0)\})$ and $\mathrm{preim}_\pi(\{\gamma(1)\})$. It is injective since there is a unique horizontal lift of γ through each point $p \in \mathrm{preim}_\pi(\{\gamma(0)\})$, and horizontal lifts through different points do not intersect. It is surjective since for each $q \in \mathrm{preim}_\pi(\{\gamma(1)\})$ we can find a p such that $q = \gamma^\uparrow_p(1)$ as follows. Let $\widetilde{p} \in \mathrm{preim}_\pi(\{\gamma(0)\})$. Then $\gamma^\uparrow_{\widetilde{p}}(1)$ belongs to the same fibre as q and hence there exists a unique $g \in G$ such that $q = \gamma^\uparrow_{\widetilde{p}}(1) \lhd g$. Recall that

$$\gamma^\uparrow_{\widetilde{p}}(\lambda) = (\sigma \circ \gamma)(\lambda) \lhd (\mathrm{P}\exp(\cdots)g_0)$$

where g_0 is the unique $g_0 \in G$ such that $\widetilde{p} = (\sigma \circ \gamma)(0) \lhd g_0$. Define $p \in \mathrm{preim}_\pi(\{\gamma(0)\})$ by

$$p := \widetilde{p} \lhd g = (\sigma \circ \gamma)(0) \lhd (g_0 \bullet g).$$

Then we have

$$\begin{aligned}
\gamma^\uparrow_p(1) &= (\sigma \circ \gamma)(1) \lhd (\mathrm{P}\exp(\cdots)g_0 \bullet g) \\
&= (\sigma \circ \gamma)(1) \lhd (\mathrm{P}\exp(\cdots)g_0) \lhd g \\
&= \gamma^\uparrow_{\widetilde{p}}(1) \lhd g \\
&= q.
\end{aligned}$$

Almost everything that we have done so far transfers with ease to an associated bundle as remarked in the section of appendix "Horizontal lifts to the associated bundle".

4.5 Curvature and torsion on principal bundle

Usually, in more elementary treatments of differential geometry or general relativity, curvature and torsion are mentioned together as properties of a covariant derivative over the tangent or the frame bundle. Since we will soon define the notion of curvature on a general principal bundle equipped with a connection, one might expect that there be a general definition of torsion on a principal bundle with a connection. However, this is not the case. Torsion requires additional structure beyond that induced by a connection. The reason why curvature and torsion are sometimes presented together is that frame bundles are already equipped, in a canonical way, with the extra structure required to define torsion.

4.5.1 Covariant exterior derivative and curvature

Since the importance of the subject, definitions and a proposition from the appendix are required:

Definition. Let (P, π, M) be a principal G-bundle with connection one-form ω. Let ϕ be a k-form (i.e. an anti-symmetric, $\mathcal{C}^\infty(P)$-multilinear map) with values in some module V. Then then *exterior covariant derivative* of ϕ is

$$\begin{aligned} \mathrm{D}\phi\colon \quad & \Gamma(TP)^{\times(k+1)} \to V \\ & (X_1, \dots, X_{k+1}) \mapsto \mathrm{d}\phi(\mathrm{hor}(X_1), \dots, \mathrm{hor}(X_{k+1})). \end{aligned}$$

Definition. Let (P, π, M) be a principal G-bundle with connection one-form ω. The *curvature* of the connection one-form ω is the Lie-algebra-valued 2-form on P

$$\Omega\colon \Gamma(TP) \times \Gamma(TP) \to T_eG$$

defined by

$$\Omega := \mathrm{D}\omega.$$

For calculation purposes, we would like to make this definition a bit more explicit.

Proposition 4.46. *Let ω be a connection one-form and Ω its curvature. Then*

$$\Omega = \mathrm{d}\omega + \omega \mathbin{\bar{\wedge}} \omega \qquad (\star)$$

with the second term on the right hand side defined as

$$(\omega \mathbin{\bar{\wedge}} \omega)(X, Y) := [\![\omega(X), \omega(Y)]\!]$$

where $X, Y \in \Gamma(TP)$ and the double bracket denotes the Lie bracket on T_eG.

Remark 4.47. If G is a matrix Lie group, and hence T_eG is an algebra of matrices of the same size as those of G, then we can write

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j.$$

We would now like to relate the curvature on a principal bundle to (local) objects on the base manifold, just like we have done for the connection one-form. Recall that a connection one-form on a principal G-bundle (P, π, M) is a T_eG-valued one-form ω on P. By using the notation $\Omega^1(P) \otimes T_eG$ for the collection (in fact, bundle) of all T_eG-valued one-forms, we have $\omega \in \Omega^1(P) \otimes T_eG$. If $\sigma \in \Gamma(TU)$ is a local section on M, we defined the Yang-Mills field $\omega^U \in \Omega^1(U) \otimes T_eG$ by pulling ω back along σ.

Definition. Let (P, π, M) be a principal G-bundle and let Ω be the curvature associated to a connection one-form on P. Let $\sigma \in \Gamma(TU)$ be a local section on M. Then, the two-form

$$\mathrm{Riem} \equiv F := \sigma^*\Omega \in \Omega^2(U) \otimes T_eG$$

is called the *Yang-Mills field strength*.

Remark 4.48. Observe that the equation $\Omega = \mathrm{d}\omega + \omega \curlywedge \omega$ on P immediately gives

$$\begin{aligned} \sigma^*\Omega &= \sigma^*(\mathrm{d}\omega + \omega \curlywedge \omega) \\ &= \sigma^*(\mathrm{d}\omega) + \sigma^*(\omega \curlywedge \omega) \\ &= \mathrm{d}(\sigma^*\omega) + \sigma^*\omega \curlywedge \sigma^*\omega. \end{aligned}$$

Since Riem is a two-form, we can write

$$\mathrm{Riem}_{\mu\nu} = (\mathrm{d}\omega^U)_{\mu\nu} + \omega^U_\mu \curlywedge \omega^U_\nu .$$

In the case of a matrix Lie group, by writing ${\Gamma^i}_{j\mu} := {(\omega^U)^i}_{j\mu}$, we can further express this in components as

$${\mathrm{Riem}^i}_{j\mu\nu} = \partial_\nu {\Gamma^i}_{j\mu} - \partial_\mu {\Gamma^i}_{j\nu} + {\Gamma^i}_{k\mu}{\Gamma^k}_{j\nu} - {\Gamma^i}_{k\nu}{\Gamma^k}_{j\mu}$$

from which we immediately observe that Riem is symmetric in the last two indices, i.e.

$${\mathrm{Riem}^i}_{j[\mu\nu]} = 0.$$

Theorem 4.49 (First Bianchi identity)**.** *Let Ω be the curvature two-form associated to a connection one-form ω on a principal bundle. Then*

$$\mathrm{D}\Omega = 0.$$

Remark 4.50. Note that since $\Omega = \mathrm{D}\omega$, Bianchi's identity can be rewritten as $\mathrm{D}^2\Omega = 0$. However, unlike the exterior derivative d, the covariant exterior derivative does *not* satisfy $\mathrm{D}^2 = 0$ in general.

4.5.2 Torsion

As discussed before, to define torsion, additional structure is required. A solder form provides an identification of V with each tangent space of M.

Remark 4.51. We can now see that the "extra structure" required to define the torsion is a choice of solder form. The following example shows that there is a canonical choice of such a form on any frame bundle bundle.

Example 4.52. Consider the frame bundle (LM, π, M) and define

$$\begin{aligned} \theta \colon \Gamma(T(LM)) &\to \mathbb{R}^{\dim M} \\ X &\mapsto (u_{\pi(X)}^{-1} \circ \pi_*)(X) \end{aligned}$$

where for each $e := (e_1, \ldots, e_{\dim M}) \in LM$, u_e is defined as

$$\begin{aligned} u_e \colon \qquad \mathbb{R}^{\dim M} &\xrightarrow{\sim} T_{\pi(e)}M \\ (x^1, \ldots, x^{\dim M}) &\mapsto x^i e_i. \end{aligned}$$

To describe the inverse map u_e^{-1} explicitly, note that to every frame $(e_1, \ldots, e_{\dim M}) \in LM$, there exists a co-frame $(f^1, \ldots, f^{\dim M}) \in L^*M$ such that

$$\begin{aligned} u_e^{-1} \colon T_{\pi(e)}M &\xrightarrow{\sim} \mathbb{R}^{\dim M} \\ Z &\mapsto (f^1(Z), \ldots, f^{\dim M}(Z)). \end{aligned}$$

Now we are ready to report the definition of torsion from the appendix:

Definition. Let (P, π, M) be a principal G-bundle with connection one-form ω and let $\theta \in \Omega^1(P) \otimes V$ be a solder form on P. Then

$$\Theta := \mathrm{D}\theta \in \Omega^2(P) \otimes V$$

is the *torsion* of ω with respect to θ.

We would like to have a similar formula for Θ as we had for Ω. However, since Θ and θ are both V-valued but ω is T_eG-valued, the term $\omega \mathbin{\wedge\!\!\!\wedge} \theta$ would be meaningless. What we have, instead, is the following

$$\Theta = \mathrm{d}\theta + \omega \mathbin{\wedge\!\!\!\cdot} \theta,$$

where the half-double wedge symbol intuitively indicates that we let ω act on θ. More precisely, in the case of a matrix Lie group, recalling that $\dim G = \dim T_eG = \dim V$, we have

$$\Theta^i = \mathrm{d}\theta^i + \omega^i{}_k \wedge \theta^k.$$

Remark 4.53. Like connection one-forms and curvatures two-forms, a torsion two-form Θ can also be pulled back to the base manifold along a local section σ as $T := \sigma^*\Theta$. In fact, *this* is the torsion that one typically meets in general relativity.

4.6 Covariant derivatives

Recall that if F is a vector space and (P, π, M) a principal G-bundle equipped with a connection, we can use the parallel transport on the associated bundle (P_F, π_F, M) and the vector space structure of F to define the differential quotient of a local section $\sigma\colon U \to P_F$ along an integral curve of some tangent vector $X \in TU$. This then allowed us to define the covariant derivative of σ at the point $\pi(X) \in U$ in the direction of $X \in TU$.

This approach to the concept of covariant derivative is very intuitive and geometric, but it is a disaster from a technical point of view as it is quite difficult to implement. There is, in fact, a neater approach to covariant differentiation.

There is an equivalence of local sections of associated bundles and equivariant functions on principal bundles.

In other words, We can now specialize to the case where F is a vector space, and hence we can require the left action $G \triangleright \colon F \xrightarrow{\sim} F$ to be linear actions on associated vector fibre bundles.

4.6.1 Construction of covariant derivatives

We now wish to construct a covariant derivative, i.e. an "operator" ∇ such that for any local section $\sigma\colon U \subseteq M \to P_F$ and any $X \in T_mU$ with $m \in U$, we have that $\nabla_X\sigma$ is again a local section $U \to P_F$ and

i) $\nabla_{fX+Y}\sigma = f\nabla_X\sigma + \nabla_Y\sigma$

ii) $\nabla_X(\sigma + \tau) = \nabla_X\sigma + \nabla_X\tau$

iii) $\nabla_X f\sigma = X(f)\sigma + f\nabla_X\sigma$

for any sections $\sigma, \tau\colon U \to P_F$, any $f \in \mathcal{C}^\infty(U)$, and any $X, Y \in T_mU$.

These (together with $\nabla_X f := X(f)$) are usually presented as the defining properties of the covariant derivative in more elementary treatments.

Recall that functions are a special case of forms, namely the 0-forms, and hence the exterior covariant derivative a function $\phi\colon P \to F$ is

$$\mathrm{D}\phi := \mathrm{d}\phi \circ \mathrm{hor}\,.$$

Proposition 4.54. *Let $\phi\colon P \to F$ be G-equivariant and let $X \in T_pP$. Then*

$$\mathrm{D}\phi(X) = \mathrm{d}\phi(X) + \omega(X) \triangleright \phi$$

Hence, it is clear from this proposition that $\mathrm{D}\phi(X)$, which we can also write as $\mathrm{D}_X\phi$, is $\mathcal{C}^\infty(P)$-linear in the X-slot, additive in the ϕ-slot and satisfies property iii) above. However, it also clearly *not* a covariant derivative since $X \in TP$ rather than $X \in TM$ and ϕ is a G-equivariant function $P \to F$ rather than a local section o (P_F, π_F, M).

We can obtain a covariant derivative from D by introducing a local trivialization on the bundle (P, π, M). Indeed, let $s\colon U \subseteq M \to P$ be a local section. Then, we can pull back the following objects

$$
\begin{aligned}
\phi\colon P \to F \qquad &\rightsquigarrow \qquad s^*\phi := \phi \circ s\colon U \to P_F \\
\omega \in \Omega^1(M) \otimes T_eG \qquad &\rightsquigarrow \qquad \omega^U := s^*\omega \in \Omega^1(U) \otimes T_eG \\
\mathrm{D}\phi \in \Omega^1(M) \otimes F \qquad &\rightsquigarrow \qquad s^*(\mathrm{D}\phi) \in \Omega^1(U) \otimes F.
\end{aligned}
$$

It is, in fact, for this last object that we will be able to define the covariant derivative. Let $X \in TU$. Then

$$
\begin{aligned}
(s^*\mathrm{D}\phi)(X) &= s^*(\mathrm{d}\phi + \omega \rhd \phi)(X) \\
&= s^*(\mathrm{d}\phi)(X) + s^*(\omega \rhd \phi)(X) \\
&= \mathrm{d}(s^*\phi)(X) + s^*(\omega)(X) \rhd s^*\phi \\
&= \mathrm{d}\sigma(X) + \omega^U(X) \rhd \sigma
\end{aligned}
$$

where we renamed $s^*\phi =: \sigma$. In summary, we can write

$$
\nabla_X \sigma = \mathrm{d}\sigma(X) + \omega^U(X) \rhd \sigma
$$

One can check that this satisfies all the properties that we wanted a covariant derivative to satisfy. Of course, we should note that this is a local definition.

Remark 4.55. Observe that the definition of covariant derivative depends on two choices which can be made quite independently of each other, namely, the choice of connection one-form ω (which determines ω^U) and the choice of linear left action $\rhd$ on F.

4.6.2 Application

Looking at quantum mechanics in curved space we may forget the basic ideas found in books and start from a differential geometric prospective. Locally what we know about QM must be preserved and we shall not modify those definitions, but we will start with the following question: what if the wave function is considered as a section of a complex valued vector bundle over $\mathbb{R}^d$? For a 1-dim line, we can say that at every point there exists a fiber, that is isomorphic to some complex vector space. Locally the wave function Ψ is just a section of the bundle total space E. The wave functions are locally sections from $\mathbb{R}^d$ to $\mathbb{C}$. If we consider now the bundle E as being an associated bundle then the principal bundle of interest if we are dealing with general changes of coordinates, is the frame bundle, with a choice of a linear left action. Then an arbitrary change of coordinates on the base manifold, will induce a corresponding change in the coordinate induced frame. The objective is to see how changing coordinates gives a change in the frames. If E is the associated bundle to the frame bundle LM over $\mathbb{R}^d$, then it is possible to establish on the frame bundle a connection. If you have a connection, you can define a covariant derivative on sections of any associated bundle. Therefore a covariant derivative acting. So saying that Ψ is a section of this associated bundle E, with fiber $\mathbb{C}$, is the transition to saying Ψ is a complex valued function, but on the total space of the principal bundle. Since we are on the principle bundle we can use there the covariant exterior derivative of Ψ, $D\Psi = \Omega$. The Ω acts from the left of Ψ, the left action of the GL(d,$\mathbb{R}$) group, that I need to establish on the fibers in order to make this an associated bundle.

The idea is to make use of the definition of covariant derivative, modify the definition:

$$\mathrm{D}\Psi(X) = \mathrm{d}\Psi(X) + \omega(X) \triangleright \Psi$$

The partial derivative alone supposes there is a complex valued function, but since we defined it to be a section on a bundle, we have extra terms coming from the definition of covariant derivative. This is what will make quantum mechanics on curved space in 2-dimension possible, even if different systems from the Cartesian are used. Defining the self-adjoint momentum operator with the notion of covariant derivative on a section of an associated bundle as, $P_\alpha = -i\nabla_\alpha$. To remind the reader, When a covariant derivative acts on a function, it acts exactly as a partial derivative, let X be a vector in the base space, then $\nabla_X f = Xf$. When it acts on a section of a bundle $\nabla_\alpha \sigma = \partial_\alpha \sigma + \omega_\alpha \sigma$, it carries the extra structure needed.

Conclusion

In the first chapter we discussed few key players in the development of the first field theory up until its generalization to a non-Abelian gauge theory. In chapter two starting with an Abelian theory we showed the relation between the gauge degree of freedom in EM to be related with a local phase degree of freedom in QM. Furthermore we discussed two fundamental relations between Mathematics and Physics, the mathematical concept of a connection 1-form and curvature 2-form with the physical counter parts of vector potential and field strength tensor, respectively. In chapter three we briefly touched upon the non-Abelian case highlighting the common futures with the Abelian counter part and account for the richer structure. In chapter four we started with a basic review of topological manifolds, bundles, product bundles, fiber bundles and principal fiber bundles. According to James Owen Weatherall [10], one might describe general relativity as the theory according to which spacetime is curved, where the curvature depends on the distribution of energy and momentum, with gravitational effects as manifestation of this spacetime curvature. If one would follow Maxwell though, then it would be necessary to define the curvature of spacetime as the result of a pull-back to the base manifold of the covariant exterior derivative acting on a connection 1-form. At the same level of description, the picture of Yang-Mills theory that one ends up with, is that the matter in the universe, (electrons, quarks, neutrinos,...) seem to have degrees of freedom at each point that are well-represented by elements of a vector space. Taken together, the spaces of local degrees of freedom form a vector bundle over spacetime, sections of which are (generalized) "matter fields". Likewise, matter fields may be associated with other generalized fields on spacetime, again construed as sections of vector bundles, representing properties matter may have, such as velocity, electromagnetic charge, color charge, isospin, etc. The reality is much more subtle and even the definition of field fails and a generalization of the notion of vector space over a field is needed by introducing modules over rings. Mathematics, just like Physics, is not enough by itself. A true revolution will happen as soon as a genuine marriage between the two disciplines will be achieved and both will be able to benefit from the result. Hilbert proposed to unify Mathematics, that dream hasn't been realized yet and a unification of Physics is even farther from happening. A century has passed and more points of contact have been found, but for the author progress will involve a coordinated motion in parallel of both disciplines. One will not stand on its own without the aid of the other. An apology to the reader is due, for using so many words instead of equations, but exiting a bachelor in physics the author feels like Faraday must have felt: *"...I am unfortunate in a want to mathematical knowledge and power of entering with facility any abstract reasoning. I am obliged to feel my way by facts placed closely together..."* The hope is that through a careful review of the basic assumptions on what we mean by space, time and spin, their structures and their representatives, some previously thought certainties, can be singled out and understood more broadly. To keep the analogy going, and by apologizing now to Faraday for even imagining to be like him, the author feels that a possible path would be to find the new Thomson that has selected the key mathematical concept and try to be a new Maxwell, that can put words into equations. To end, a quote from Hermann

Weyl about the orthogonal transformations as automorphisms of Euclidean vector spaces: *"...only with the spinors do we strike that level in the theory of its representations on which Euclid himself, flourishing ruler and compass, so deftly moves in the realm of geometric figures. In some way Euclid's geometry must be deeply connected with the existence of the spin representation."*

Appendices

A Topological manifolds and bundles

A.1 Topological manifolds

Definition. A paracompact, Hausdorff, topolog ical space $(M, \mathcal{O})$ is called a *d-dimensional (topological) manifold* if for every point $p \in M$ there exist a neighbourhood $U(p)$ and a homeomorphism $x\colon U(p) \to x(U(p)) \subseteq \mathbb{R}^d$. We also write $\dim M = d$.

Intuitively, a d-dimensional manifold is a topological space which locally (i.e. around each point) looks like $\mathbb{R}^d$. Note that, strictly speaking, what we have just defined are *real* topological manifolds. We could define *complex* topological manifolds as well, simply by requiring that the map x be a homeomorphism onto an open subset of $\mathbb{C}^d$.

Proposition .56. *Let M be a d-dimensional manifold and let $U, V \subseteq M$ be open, with $U \cap V \neq \varnothing$. If x and y are two homeomorphisms*

$$x\colon U \to x(U) \subseteq \mathbb{R}^d \qquad \textit{and} \qquad y\colon V \to y(V) \subseteq \mathbb{R}^{d'},$$

then $d = d'$.

This ensures that the concept of dimension is indeed well-defined, i.e. it is the same at every point, at least on each connected component of the manifold.

Definition. Let $(M, \mathcal{O})$ be a topological manifold and let $N \subseteq M$. Then $(N, \mathcal{O}|_N)$ is called a *submanifold* of $(M, \mathcal{O})$ if it is a manifold in its own right.

Definition. Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological manifolds of dimension m and n, respectively. Then, $(M \times N, \mathcal{O}_{M\times N})$ is a topological manifold of dimension $m + n$ called the *product manifold*.

Example .57. We have $T^2 = S^1 \times S^1$ not just as topological spaces, but as topological manifolds as well. This is a special case of the n-torus:

$$T^n := \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}},$$

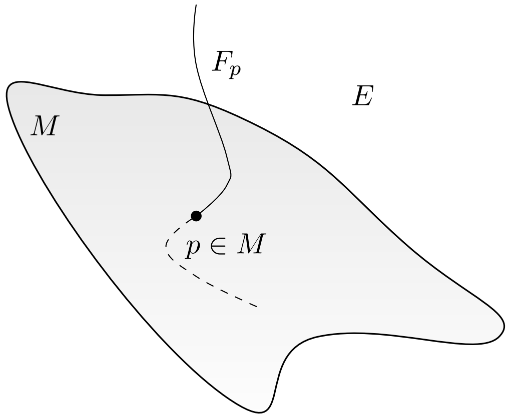
which is an n-dimensional manifold.

A.2 Bundles

Definition. A *bundle* (of topological manifolds) is a triple (E, π, M) where E and M are topological manifolds called the *total space* and the *base space* respectively, and π is a continuous, surjective map $\pi\colon E \to M$ called the *projection map*.

We will often denote the bundle (E, π, M) by $E \xrightarrow{\pi} M$.

Definition. Let $E \xrightarrow{\pi} M$ be a bundle and let $p \in M$. Then, $F_p := \mathrm{preim}_\pi(\{p\})$ is called the *fibre* at the point p.



Intuitively, the fibre at the point $p \in M$ is a set of points in E (represented below as a line) attached to the point p. The projection map sends all the points in the fibre F_p to the point p.

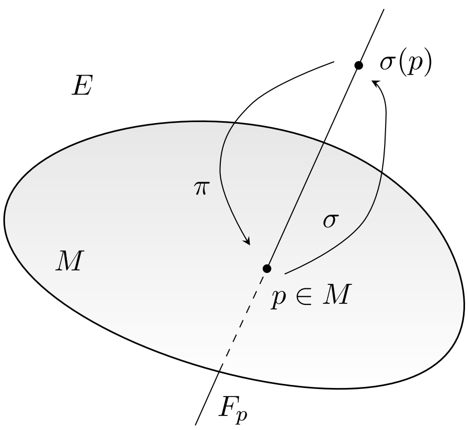
Definition. Let $E \xrightarrow{\pi} M$ be a bundle and let F be a manifold. Then, $E \xrightarrow{\pi} M$ is called a *fibre bundle*, with (typical) fibre F, if:

$$\forall\, p \in M : \operatorname{preim}_\pi(\{p\}) \cong_{\text{top}} F.$$

A fibre bundle is often represented diagrammatically as:

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \big\downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

Definition. Let $E \xrightarrow{\pi} M$ be a bundle. A map $\sigma\colon M \to E$ is called a *(cross-)section* of the bundle if $\pi \circ \sigma = \mathrm{id}_M$.



Definition. A *sub-bundle* of a bundle (E, π, M) is a triple (E', π', M') where $E' \subseteq E$ and $M' \subseteq M$ are submanifolds and $\pi' := \pi|_{E'}$.

Definition. Let (E, π, M) be a bundle and let $N \subseteq M$ be a submanifold. The *restricted bundle* (to N) is the triple (E, π', N) where:

$$\pi' := \pi|_{\mathrm{preim}_\pi(N)}$$

Definition. Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ be bundles and let $u\colon E \to E'$ and $v\colon M \to M'$ be maps. Then (u, v) is called a *bundle morphism* if the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{v} & M' \end{array}$$

i.e. if $\pi' \circ u = v \circ \pi$.

If (u, v) and (u, v') are both bundle morphisms, then $v = v'$. That is, given u, if there exists v such that (u, v) is a bundle morphism, then v is unique.

Definition. Two bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ are said to be *isomorphic (as bundles)* if there exist bundle morphisms (u, v) and (u^{-1}, v^{-1}) satisfying:

$$\begin{array}{ccc} E & \underset{u^{-1}}{\overset{u}{\rightleftarrows}} & E' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ M & \underset{v^{-1}}{\overset{v}{\rightleftarrows}} & M' \end{array}$$

Such a (u, v) is called a *bundle isomorphism* and we write $E \xrightarrow{\pi} M \cong_{\mathrm{bdl}} E' \xrightarrow{\pi'} M'$.

Definition. A bundle $E \xrightarrow{\pi} M$ is said to be *locally isomorphic (as a bundle)* to a bundle $E' \xrightarrow{\pi'} M'$ if for all $p \in M$ there exists a neighbourhood $U(p)$ such that the restricted bundle:

$$\mathrm{preim}_\pi(U(p)) \xrightarrow{\pi|_{\mathrm{preim}_\pi(U(p))}} U(p)$$

is isomorphic to the bundle $E' \xrightarrow{\pi'} M'$.

Definition. A bundle $E \xrightarrow{\pi} M$ is said to be:

i) *trivial* if it is isomorphic to a product bundle;

ii) *locally trivial* if it is locally isomorphic to a product bundle.

Definition. Let $E \xrightarrow{\pi} M$ be a bundle and let $f\colon M' \to M$ be a map from some manifold M'. The *pull-back bundle of* $E \xrightarrow{\pi} M$ *induced by* f is defined as $E' \xrightarrow{\pi'} M'$, where:

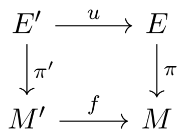
$$E' := \{(m', e) \in M' \times E \mid f(m') = \pi(e)\}$$

and $\pi'(m', e) := m'$.

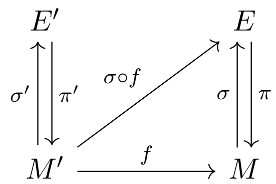
If $E' \xrightarrow{\pi'} M'$ is the pull-back bundle of $E \xrightarrow{\pi} M$ induced by f, then one can easily construct a bundle morphism by defining:

$$
\begin{aligned}
u \colon \quad E' &\to E \\
(m', e) &\mapsto e
\end{aligned}
$$

This corresponds to the diagram:



Remark .58. Sections on a bundle pull back to the pull-back bundle. Indeed, let $E' \xrightarrow{\pi'} M'$ be the pull-back bundle of $E \xrightarrow{\pi} M$ induced by f.



If σ is a section of $E \xrightarrow{\pi} M$, then $\sigma \circ f$ determines a map from M' to E which sends each $m' \in M'$ to $\sigma(f(m')) \in E$. However, since σ is a section, we have:

$$\pi(\sigma(f(m')) = (\pi \circ \sigma \circ f)(m') = (\mathrm{id}_M \circ f)(m') = f(m')$$

and hence $(m', (\sigma \circ f)(m')) \in E'$ by definition of E'. Moreover:

$$\pi'(m', (\sigma \circ f)(m')) = m'$$

and hence the map:

$$
\begin{aligned}
\sigma' \colon M' &\to E' \\
m' &\mapsto (m', (\sigma \circ f)(m'))
\end{aligned}
$$

satisfies $\pi' \circ \sigma' = \mathrm{id}_{M'}$ and it is thus a section on the pull-back bundle $E' \xrightarrow{\pi'} M'$.

A.3 Viewing manifolds from atlases

Definition. Let $(M, \mathcal{O})$ be a d-dimensional manifold. Then, a pair (U, x) where $U \in \mathcal{O}$ and $x \colon U \to x(U) \subseteq \mathbb{R}^d$ is a homeomorphism, is said to be a *chart* of the manifold.

The *component functions (or maps)* of $x \colon U \to x(U) \subseteq \mathbb{R}^d$ are the maps:

$$
\begin{aligned}
x^i \colon U &\to \mathbb{R} \\
p &\mapsto \mathrm{proj}_i(x(p))
\end{aligned}
$$

for $1 \leq i \leq d$, where $\mathrm{proj}_i(x(p))$ is the i-th component of $x(p) \in \mathbb{R}^d$. The $x^i(p)$ are called the *co-ordinates* of the point $p \in U$ with respect to the chart (U, x).

Definition. An *atlas* of a manifold M is a collection $\mathscr{A} := \{(U_\alpha, x_\alpha) \mid \alpha \in \mathcal{A}\}$ of charts such that:

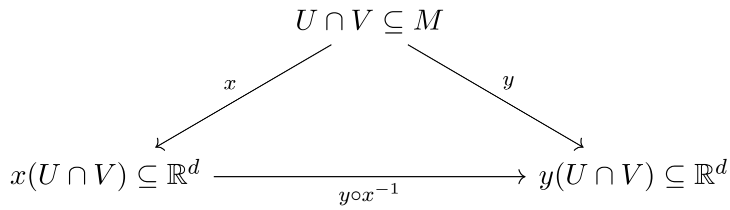
$$\bigcup_{\alpha\in\mathcal{A}} U_\alpha = M.$$

Definition. Two charts (U, x) and (V, y) are said to be $\mathcal{C}^0$-*compatible* if either $U \cap V = \varnothing$ or the map:

$$y \circ x^{-1} \colon x(U \cap V) \to y(U \cap V)$$

is continuous.

Note that $y \circ x^{-1}$ is a map from a subset of $\mathbb{R}^d$ to a subset of $\mathbb{R}^d$.



Since the maps x and y are homeomorphisms, the composition map $y \circ x^{-1}$ is also a homeomorphism and hence continuous. Therefore, any two charts on a topological manifold are $\mathcal{C}^0$-compatible. This definition my thus seem redundant since it applies to every pair of charts. However, it is just a "warm up" since we will later refine this definition and define the *differentiability* of maps on a manifold in terms of $\mathcal{C}^k$-compatibility of charts.

Remark .59. The map $y \circ x^{-1}$ (and its inverse $x \circ y^{-1}$) is called the *co-ordinate change map* or *chart transition map*.

Definition. A $\mathcal{C}^0$-*atlas* of a manifold is an atlas of pairwise $\mathcal{C}^0$-compatible charts.

Note that any atlas is also a $\mathcal{C}^0$-*atlas*.

Definition. A $\mathcal{C}^0$-atlas $\mathscr{A}$ is said to be a *maximal atlas* if for every $(U, x) \in \mathscr{A}$, we have $(V, y) \in \mathscr{A}$ for all (V, y) charts that are $\mathcal{C}^0$-compatible with (U, x).

Example .60. Not every $\mathcal{C}^0$-atlas is a maximal atlas. Indeed, consider $(\mathbb{R}, \mathcal{O}_{\text{std}})$ and the atlas $\mathscr{A} := (\mathbb{R}, \text{id}_\mathbb{R})$. Then $\mathscr{A}$ is not maximal since $((0, 1), \text{id}_\mathbb{R})$ is a chart which is $\mathcal{C}^0$-compatible with $(\mathbb{R}, \text{id}_\mathbb{R})$ but $((0, 1), \text{id}_\mathbb{R}) \notin \mathscr{A}$.

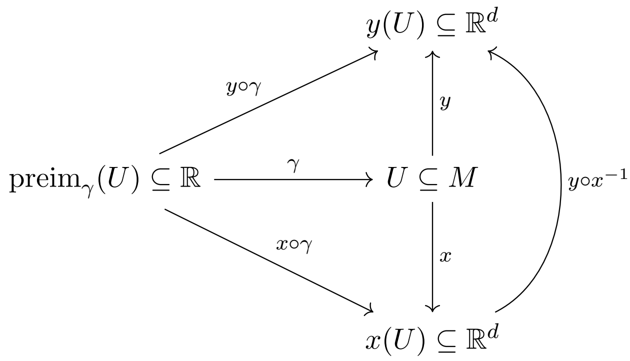
We can now look at "objects on" topological manifolds from two points of view. For instance, consider a curve on a d-dimensional manifold M, i.e. a map $\gamma \colon \mathbb{R} \to M$. We now ask whether this curve is continuous, as it should be if models the trajectory of a particle on the "physical space" M.

A first answer is that $\gamma \colon \mathbb{R} \to M$ is continuous if it is continuous as a map between the topological spaces $\mathbb{R}$ and M.

However, the answer that may be more familiar to you from undergraduate physics is the following. We consider only a portion (open subset U) of the physical space M and, instead of studying the map $\gamma \colon \text{preim}_\gamma(U) \to U$ directly, we study the map:

$$x \circ \gamma \colon \text{preim}_\gamma(U) \to x(U) \subseteq \mathbb{R}^d,$$

where (U, x) is a chart of M. More likely, you would be checking the continuity of the co-ordinate maps $x^i \circ \gamma$, which would then imply the continuity of the "real" curve $\gamma\colon \text{preim}_\gamma(U) \to U$ (real, as opposed to its co-ordinate representation).



At some point you may wish to use a different "co-ordinate system" to answer a different question. In this case, you would chose a different chart (U, y) and then study the map $y \circ \gamma$ or its co-ordinate maps. Notice however that some results (e.g. the continuity of γ) obtained in the previous chart (U, x) can be immediately "transported" to the new chart (U, y) via the chart transition map $y \circ x^{-1}$. Moreover, the map $y \circ x^{-1}$ allows us to, intuitively speaking, forget about the inner structure (i.e. U and the maps γ, x and x) which, in a sense, is the real world, and only consider $\text{preim}_\gamma(U) \subseteq \mathbb{R}$ and $x(U), y(U) \subseteq \mathbb{R}^d$ together with the maps between them, which is our representation of the real world.

B Construction of the tangent bundle

B.1 Cotangent spaces and the differential

Since the tangent space is a vector space, we can do all the constructions we saw previously in the abstract vector space setting.

Definition. Let M be a manifold and $p \in M$. The *cotangent space* to M at p is

$$T_p^* M := (T_p M)^*.$$

Since $\dim T_p M$ is finite, we have $T_p M \cong_{\mathrm{vec}} T_p^* M$. If $\{(\frac{\partial}{\partial x^a})_p\}$ is the basis of $T_p M$ induced by some chart (U, x), then the dual basis is denoted as $\{(\mathrm{d}x^a)_p\}$. We have, by definition

$$(\mathrm{d}x^a)_p \left(\left(\frac{\partial}{\partial x^b} \right)_p \right) = \delta^a_b.$$

Once we have the cotangent space, we can define the tensor spaces.

Definition. Let M be a manifold and $p \in M$. The *tensor space* $(T^r_s)_p M$ is defined as

$$(T^r_s)_p M := T^r_s(T_p M) = \underbrace{T_p M \otimes \cdots \otimes T_p M}_{r \text{ copies}} \otimes \underbrace{T_p^* M \otimes \cdots \otimes T_p^* M}_{s \text{ copies}}.$$

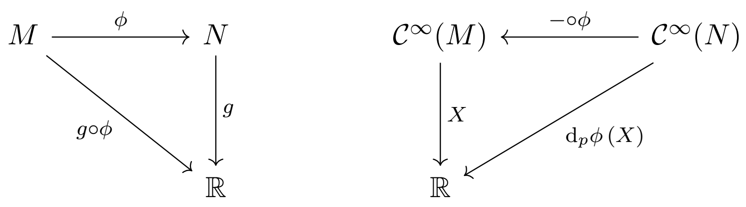
Definition. Let M and N be manifolds and let $\phi\colon M \to N$ be smooth. The *differential* (or *derivative*) *of* ϕ *at* $p \in M$ is the linear map

$$\begin{aligned} \mathrm{d}_p\phi\colon T_p M &\xrightarrow{\sim} T_{\phi(p)} N \\ X &\mapsto \mathrm{d}_p\phi\,(X) \end{aligned}$$

where $\mathrm{d}_p\phi\,(X)$ is the tangent vector to N at $\phi(p)$

$$\begin{aligned} \mathrm{d}_p\phi\,(X)\colon \mathcal{C}^\infty(N) &\xrightarrow{\sim} \mathbb{R} \\ g &\mapsto (\mathrm{d}_p\phi\,(X))(g) := X(g \circ \phi). \end{aligned}$$

If this definition looks confusing, it is worth it to pause and think about what it is saying. Intuitively, if ϕ takes us from M to N, then $\mathrm{d}_p\phi$ takes us from $T_p M$ to $T_{\phi(p)} N$. The way in which it does so, is the following.



Given $X \in T_p M$, we want to construct $\mathrm{d}_p\phi\,(X) \in T_{\phi(p)} N$, i.e. a derivation on N at $f(p)$. Derivations act on functions. So, given $g\colon N \to \mathbb{R}$, we want to construct a real number by using ϕ and X. There is really only one way to do is. If we precompose g with ϕ, we obtain $g \circ \phi\colon M \to \mathbb{R}$, which is an element of $\mathcal{C}^\infty(M)$. We can then happily apply X to this function to obtain a real number. You should check that $\mathrm{d}_p\phi\,(X)$ is indeed a tangent vector to N.

Remark .61. Note that, to be careful, we should replace $\mathcal{C}^\infty(M)$ and $\mathcal{C}^\infty(N)$ above with $\mathcal{C}^\infty(U)$ and $\mathcal{C}^\infty(V)$, where $U \subseteq M$ and $V \subseteq N$ are open and contain p and $\phi(p)$, respectively.

Example .62. If $M = \mathbb{R}^d$ and $N = \mathbb{R}^{d'}$, then the differential of $f\colon \mathbb{R}^d \to \mathbb{R}^{d'}$ at $p \in \mathbb{R}^d$

$$\mathrm{d}_p f\colon T_p\mathbb{R}^d \cong_{\text{vec}} \mathbb{R}^d \to T_{f(p)}\mathbb{R}^{d'} \cong_{\text{vec}} \mathbb{R}^{d'}$$

is none other than the Jacobian of f at p.

A special case of the differential is the gradient of a function in $\mathcal{C}^\infty(M)$.

Definition. Let M be a manifold and let $f\colon M \to \mathbb{R}$ be smooth. The *gradient of f at $p \in M$* is the covector

$$\begin{aligned} \mathrm{d}_p f\colon T_pM &\xrightarrow{\sim} T_{f(p)}\mathbb{R} \cong_{\text{vec}} \mathbb{R} \\ X &\mapsto \mathrm{d}_p f(X) := X(f). \end{aligned}$$

In fact, we can define the *gradient operator at $p \in M$* as the $\mathbb{R}$-linear map

$$\begin{aligned} \mathrm{d}_p\colon \mathcal{C}^\infty(U) &\xrightarrow{\sim} T_p^*M \\ f &\mapsto \mathrm{d}_p f, \end{aligned}$$

with $p \in U \subseteq M$.

Remark .63. Note that, by writing $\mathrm{d}_p f(X) := X(f)$, we have committed a slight (but nonetheless real) abuse of notation. Since $\mathrm{d}_p f(X) \in T_{f(p)}\mathbb{R}$, it takes in a function and return a real number, but $X(f)$ is already a real number! This is due to the fact that we have implicitly employed the isomorphism

$$\begin{aligned} \iota_d\colon T_p\mathbb{R}^d &\to \mathbb{R}^d \\ X &\mapsto (X(\mathrm{proj}_1), \ldots, X(\mathrm{proj}_d)), \end{aligned}$$

which, when $d = 1$, reads

$$\begin{aligned} \iota_1\colon T_p\mathbb{R} &\to \mathbb{R} \\ X &\mapsto X(\mathrm{id}_\mathbb{R}). \end{aligned}$$

In our case, we have

$$\mathrm{d}_p f(X) := X(- \circ f) \mapsto X(\mathrm{id}_\mathbb{R} \circ f) = X(f).$$

This notwithstanding, the best way to think of $\mathrm{d}_p f$ is as a covector, i.e. $\mathrm{d}_p f$ takes in a tangent vector X and returns the real number $X(f)$, in a linear fashion.

Recall that if (U, x) is a chart on M, then the co-ordinate maps $x^a\colon U \to x(U) \subseteq \mathbb{R}^{\dim M}$ are smooth functions on U. We can thus apply the gradient operator d_p (with $p \in U$) to each of them to obtain $(\dim M)$-many elements of T_p^*M.

Proposition .64. *Let (U, x) be a chart on M, with $p \in U$. The set $\mathcal{B} = \{\mathrm{d}_p x^a \mid 1 \leq a \leq \dim M\}$ forms a basis of T_p^*M.*

Proof. We already know that $T_p^*M = \dim M$, since it is the dual space to T_pM. As $|\mathcal{B}| = \dim M$ by construction, it suffices to show that it is linearly independent. Suppose that

$$\lambda_a \, \mathrm{d}_p x^a = 0,$$

for some $\lambda_a \in \mathbb{R}$. Applying the left hand side to the basis element $\left(\frac{\partial}{\partial x^b}\right)_p$ yields

$$\begin{aligned}
\lambda_a \, \mathrm{d}_p x^a \left(\left(\frac{\partial}{\partial x^b} \right)_p \right) &= \lambda_a \left(\frac{\partial}{\partial x^b} \right)_p (x^a) && \text{(definition of } \mathrm{d}_p x^a\text{)} \\
&= \lambda_a \, \partial_b (x^a \circ x^{-1})(x(p)) && \text{(definition of } \left(\tfrac{\partial}{\partial x^b}\right)_p\text{)} \\
&= \lambda_a \, \partial_b (\mathrm{proj}_a)(x(p)) \\
&= \lambda_a \, \delta^a_b \\
&= \lambda_b.
\end{aligned}$$

Therefore, $\mathcal{B}$ is linearly independent and hence a basis of T_p^*M. Moreover, since we have shown that

$$\mathrm{d}_p x^a \left(\left(\frac{\partial}{\partial x^b} \right)_p \right) = \delta^a_b,$$

this basis is, in fact, the dual basis to $\{\left(\frac{\partial}{\partial x^a}\right)_p\}$. □

Remark .65. Note a slight subtlety. Given a chart (U, x) and the induced basis $\{\left(\frac{\partial}{\partial x^a}\right)_p\}$ of T_pM, the dual basis to $\{\left(\frac{\partial}{\partial x^a}\right)_p\}$ exists simply by virtue of T_p^*M being the dual space to T_pM. What we have shown above is that the elements of this dual basis are given explicitly by the gradients of the co-ordinate maps of (U, x). In our notation, we have

$$(\mathrm{d}x^a)_p = \mathrm{d}_p x^a, \qquad 1 \leq a \leq \dim M.$$

B.2 Push-forward and pull-back

The push-forward of a smooth map $\phi \colon M \to N$ at $p \in M$ is just another name for the differential of ϕ at p. We give the definition again in order to establish the new notation.

Definition. Let $\phi \colon M \to N$ be a smooth map between smooth manifolds. The *push-forward* of ϕ at $p \in M$ is the linear map:

$$\begin{aligned}
(\phi_*)_p \colon T_pM &\xrightarrow{\sim} T_{\phi(p)}N \\
X &\mapsto (\phi_*)_p(X) := X(- \circ \phi).
\end{aligned}$$

If $\gamma \colon \mathbb{R} \to M$ is a smooth curve on M and $\phi \colon M \to N$ is smooth, then $\phi \circ \gamma \colon \mathbb{R} \to N$ is a smooth curve on N.

Proposition .66. *Let $\phi \colon M \to N$ be smooth. The tangent vector $X_{\gamma,p} \in T_pM$ is pushed forward to the tangent vector $X_{\phi\circ\gamma,\phi(p)} \in T_{\phi(p)}N$, i.e.*

$$(\phi_*)_p(X_{\gamma,p}) = X_{\phi\circ\gamma,\phi(p)}.$$

Proof. Let $f \in \mathcal{C}^\infty(V)$, with (V, x) a chart on N and $\phi(p) \in V$. By applying the definitions, we have

$$\begin{aligned}
(\phi_*)_p(X_{\gamma,p})(f) &= (X_{\gamma,p})(f \circ \phi) && \text{(definition of } (\phi_*)_p) \\
&= ((f \circ \phi) \circ \gamma)'(0) && \text{(definition of } X_{\gamma,p}) \\
&= (f \circ (\phi \circ \gamma))'(0) && \text{(associativity of } \circ) \\
&= X_{\phi\circ\gamma,\phi(p)}(f) && \text{(definition of } X_{\phi\circ\gamma,\phi(p)})
\end{aligned}$$

Since f was arbitrary, we have $(\phi_*)_p(X_{\gamma,p}) = X_{\phi\circ\gamma,\phi(p)}$. □

Related to the push-forward, there is the notion of pull-back of a smooth map.

Definition. Let $\phi\colon M \to N$ be a smooth map between smooth manifolds. The *pull-back* of ϕ at $p \in M$ is the linear map:

$$\begin{aligned}
(\phi^*)_p \colon T^*_{\phi(p)}N &\xrightarrow{\sim} T^*_pM \\
\omega &\mapsto (\phi^*)_p(\omega),
\end{aligned}$$

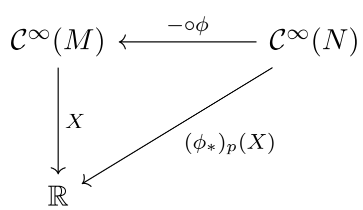
where $(\phi^*)_p(\omega)$ is defined as

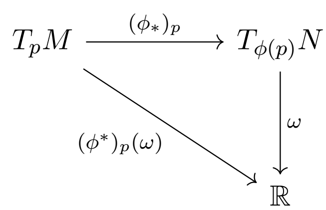
$$\begin{aligned}
(\phi^*)_p(\omega) \colon T_pM &\xrightarrow{\sim} \mathbb{R} \\
X &\mapsto \omega((\phi_*)_p(X)),
\end{aligned}$$

In words, if ω is a covector on N, its pull-back $(\phi^*)_p(\omega)$ is a covector on M. It acts on tangent vectors on M by first pushing them forward to tangent vectors on N, and then applying ω to them to produce a real number.

Remark .67. If you don't see it immediately, then you should spend some time proving that all the maps that we have defined so far and *claimed* to be linear are, in fact, linear.

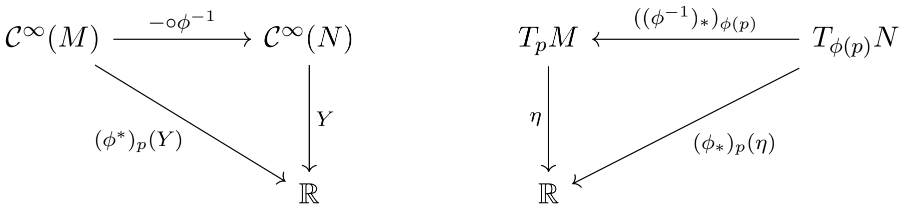
Remark .68. We have seen that, given a smooth $\phi\colon M \to N$, we can push a vector $X \in T_pM$ forward to a vector $(\phi_*)_p(X) \in T_{\phi(p)}N$, and pull a covector $\omega \in T^*_{\phi(p)}N$ back to a covector $(\phi^*)_p(\omega) \in T^*_pM$.





However, if $\phi\colon M \to N$ is a diffeomorphism, then we can also pull a vector $Y \in T_{\phi(p)}N$ back to a vector $(\phi^*)_p(Y) \in T_pM$, and push a covector $\eta \in T^*_pM$ forward to a covector $(\phi_*)_p(\eta) \in T^*_{\phi(p)}N$, by using ϕ^{-1} as follows:

$$\begin{aligned}
(\phi^*)_p(Y) &:= ((\phi^{-1})_*)_{\phi(p)}(Y) \\
(\phi_*)_p(\eta) &:= ((\phi^{-1})^*)_{\phi(p)}(\eta).
\end{aligned}$$



This is only possible if ϕ is a diffeomorphism. In general, you should keep in mind that

Vectors are pushed forward,
covectors are pulled back.

Remark .69. Given a smooth map $\phi\colon M \to N$, if $f \in \mathcal{C}^\infty(N)$, then $f \circ \phi$ is often called the pull-back of f along ϕ. Similarly, if γ is a curve on M, then $\phi \circ \gamma$ is called the push-forward of γ along ϕ. For example, we can say that

- the push-forward of a tangent vector acting on a function is the tangent vector acting on the pull-back of the function;
- the push-forward of a tangent vector to a curve is the tangent vector to the push-forward of the curve.

B.3 Immersions and embeddings

We will now consider the question of under which circumstances a smooth manifold can "sit" in $\mathbb{R}^d$, for some $d \in \mathbb{N}$. There are, in fact, two notions of sitting inside another manifold, called immersion and embedding.

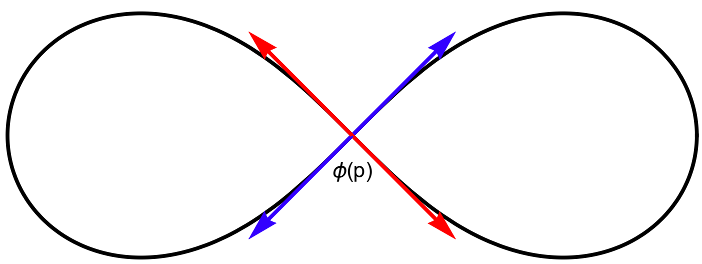
Definition. A smooth map $\phi\colon M \to N$ is said to be an *immersion* of M into N if the derivative

$$\mathrm{d}_p\phi \equiv (\phi_*)_p \colon T_pM \xrightarrow{\sim} T_{\phi(p)}N$$

is injective, for all $p \in M$. The manifold M is said to be an *immersed submanifold* of N.

From the theory of linear algebra, we immediately deduce that, for $\phi\colon M \to N$ to be an immersion, we must have $\dim M \leq \dim N$. A closely related notion is that of a *submersion*, where we require each $(\phi_*)_p$ to be surjective, and thus we must have $\dim M \geq \dim N$. However, we will not need this here.

Example .70. Consider the map $\phi\colon S^1 \to \mathbb{R}^2$ whose image is reproduced below.



The map ϕ is not injective, i.e. there are $p, q \in S^1$, with $p \neq q$ and $\phi(p) = \phi(q)$. Of course, this means that $T_{\phi(p)}\mathbb{R}^2 = T_{\phi(q)}\mathbb{R}^2$. However, the maps $(\phi_*)_p$ and $(\phi_*)_q$ are both injective, with their images being represented by the blue and red arrows, respectively. Hence, the map ϕ is immersion.

Definition. A smooth map $\phi\colon M \to N$ is said to be a *(smooth) embedding* of M into N if

- $\phi\colon M \to N$ is an immersion;
- $M \cong_{\text{top}} \phi(M) \subseteq N$, where $\phi(M)$ carries the subset topology inherited from N.

The manifold M is said to be an *embedded submanifold* of N.

Remark .71. If a continuous map between topological spaces satisfies the second condition above, then it is called a *topological embedding.* Therefore, a smooth embedding is a topological embedding which is also an immersion (as opposed to simply being a smooth topological embedding).

In the early days of differential geometry there were two approaches to study manifolds. One was the extrinsic view, within which manifolds are defined as special subsets of $\mathbb{R}^d$, and the other was the intrinsic view, which is the view that we have adopted here.

Whitney's theorem, which we will state without proof, states that these two approaches are, in fact, equivalent.

Theorem .72 (Whitney)**.** *Any smooth manifold M can be*

- *embedded in $\mathbb{R}^{2\dim M}$;*
- *immersed in $\mathbb{R}^{2\dim M-1}$.*

Example .73. The Klein bottle can be embedded in $\mathbb{R}^4$ but not in $\mathbb{R}^3$. It can, however, be immersed in $\mathbb{R}^3$.

What we have presented above is referred to as the *strong* version of Whitney's theorem. There is a weak version as well, but there are also even stronger versions of this result, such as the following.

Theorem .74. *Any smooth manifold can be immersed in $\mathbb{R}^{2\dim M - a(\dim M)}$, where $a(n)$ is the number of 1s in a binary expansion of $n \in \mathbb{N}$.*

Example .75. If $\dim M = 3$, then as

$$3_{10} = (1 \times 2^1 + 1 \times 2^0)_{10} = 11_2,$$

we have $a(\dim M) = 2$, and thus every 3-dimensional manifold can be immersed into $\mathbb{R}^4$. Note that even the strong version of Whitney's theorem only tells us that we can immerse M into $\mathbb{R}^5$.

B.4 The tangent bundle

Definition. Given a smooth manifold M, the *tangent bundle* of M is the disjoint union of all the tangent spaces to M, i.e.

$$TM := \coprod_{p \in M} T_p M,$$

equipped with the canonical projection map

$$\begin{aligned} \pi \colon TM &\to M \\ X &\mapsto p, \end{aligned}$$

where p is the unique $p \in M$ such that $X \in T_p M$.

We now need to equip TM with the structure of a smooth manifold. We can achieve this by constructing a smooth atlas for TM from a smooth atlas on M, as follows.

Let $\mathscr{A}_M$ be a smooth atlas on M and let $(U, x) \in \mathscr{A}_M$. If $X \in \operatorname{preim}_\pi(U) \subseteq TM$, then $X \in T_{\pi(X)} M$, by definition of π. Moreover, since $\pi(X) \in U$, we can expand X in terms of the basis induced by the chart (U, x):

$$X = X^a \left(\frac{\partial}{\partial x^a} \right)_{\pi(X)},$$

where $X^1, \ldots, X^{\dim M} \in \mathbb{R}$. We can then define the map

$$\begin{aligned} \xi \colon \operatorname{preim}_\pi(U) &\to x(U) \times \mathbb{R}^{\dim M} \cong_{\text{set}} \mathbb{R}^{2 \dim M} \\ X &\mapsto (x(\pi(X)), X^1, \ldots, X^{\dim M}). \end{aligned}$$

Assuming that TM is equipped with a suitable topology, for instance the initial topology (i.e. the coarsest topology on TM that makes π continuous), we claim that the pair $(\operatorname{preim}_\pi(U), \xi)$ is a chart on TM and

$$\mathscr{A}_{TM} := \{ (\operatorname{preim}_\pi(U), \xi) \mid (U, x) \in \mathscr{A}_M \}$$

is a smooth atlas on TM. Note that, from its definition, it is clear that ξ is a bijection. We will not show that $(\operatorname{preim}_\pi(U), \xi)$ is a chart here, but we will show that $\mathscr{A}_{TM}$ is a smooth atlas.

Proposition .76. *Any two charts* $(\operatorname{preim}_\pi(U), \xi), (\operatorname{preim}_\pi(\widetilde{U}), \widetilde{\xi}) \in \mathscr{A}_{TM}$ *are* $\mathcal{C}^\infty$*-compatible.*

Proof. Let (U, x) and $(\widetilde{U}, \widetilde{x})$ be the two charts on M giving rise to $(\operatorname{preim}_\pi(U), \xi)$ and $(\operatorname{preim}_\pi(\widetilde{U}), \widetilde{\xi})$, respectively. We need to show that the map

$$\widetilde{\xi} \circ \xi^{-1} \colon x(U \cap \widetilde{U}) \times \mathbb{R}^{\dim M} \to \widetilde{x}(U \cap \widetilde{U}) \times \mathbb{R}^{\dim M}$$

is smooth, as a map between open subsets of $\mathbb{R}^{2 \dim M}$. Recall that such a map is smooth if, and only if, it is smooth componentwise. On the first $\dim M$ components, $\widetilde{\xi} \circ \xi^{-1}$ acts as

$$\begin{aligned} \widetilde{x} \circ x^{-1} \colon x(U \cap \widetilde{U}) &\to \widetilde{x}(U \cap \widetilde{U}) \\ x(p) &\mapsto \widetilde{x}(p), \end{aligned}$$

while on the remaining $\dim M$ components it acts as the change of vector components we met previously, i.e.

$$X^a \mapsto \widetilde{X}^a = \partial_b(y^a \circ x^{-1})(x(p))\, X^b.$$

Hence, we have

$$\begin{aligned} \widetilde{\xi} \circ \xi^{-1} \colon \qquad & x(U \cap \widetilde{U}) \times \mathbb{R}^{\dim M} \to \widetilde{x}(U \cap \widetilde{U}) \times \mathbb{R}^{\dim M} \\ & (x(\pi(X)), X^1, \dots, X^{\dim M}) \mapsto (\widetilde{x}(\pi(X)), \widetilde{X}^1, \dots, \widetilde{X}^{\dim M}), \end{aligned}$$

which is smooth in each component, and hence smooth. □

B.5 Vector fields

Definition. Let M be a smooth manifold, and let $TM \xrightarrow{\pi} M$ be its tangent bundle. A *vector field* on M is a smooth section of the tangent bundle, i.e. a smooth map $\sigma \colon M \to TM$ such that $\pi \circ \sigma = \mathrm{id}_M$.

$$\begin{array}{c} TM \\ \sigma \uparrow \downarrow \pi \\ M \end{array}$$

We denote the set of all vector fields on M by $\Gamma(TM)$, i.e.

$$\Gamma(TM) := \{\sigma \colon M \to TM \mid \sigma \text{ is smooth and } \pi \circ \sigma = \mathrm{id}_M\}.$$

This is, in fact, the standard notation for the set of all sections on a bundle.

Definition. Let $\phi \colon M \to N$ be smooth. The *push-forward* ϕ_* is defined as

$$\begin{aligned} \phi_* \colon TM &\to TN \\ X &\mapsto (\phi_*)_{\pi(X)}(X). \end{aligned}$$

Any vector $X \in TM$ must belong to T_pM for some $p \in M$, namely $p = \pi(X)$. The map ϕ_* simply takes any vector $X \in TM$ and applies the usual push-forward at the "right" point, producing a vector in TN. One can similarly define $\phi^* \colon T^*N \to T^*M$.

The ideal next step would be to try to construct a map $\Phi_* \colon \Gamma(TM) \to \Gamma(TN)$ that allows us to push vector fields on M forward to vector fields on N. Given $\sigma \in \Gamma(TM)$, we would like to construct a $\Phi_*(\sigma) \in \Gamma(TN)$. This is not trivial, since $\Phi_*(\sigma)$ needs to be, in particular, a smooth map $N \to TN$. Note that the composition $\phi_* \circ \sigma$ is a map $M \to TN$, and hence $\mathrm{im}_{\phi_* \circ \sigma}(M) \subseteq TN$. Thus, we can try to define $\Phi_*(\sigma)$ by mapping each $p \in N$ to some tangent vector in $\mathrm{im}_{\phi_* \circ \sigma}(M)$. Unfortunately, there are at least two ways in which this can go wrong.

1. The map ϕ may fail to be injective. Then, there would be two points $p_1, p_2 \in M$ such that $p_1 \neq p_2$ and $\phi(p_1) = \phi(p_2) =: q \in N$. Hence, we would have two tangent vectors on N with base-point q, namely $(\phi_* \circ \sigma)(p_1)$ and $(\phi_* \circ \sigma)(p_2)$. These two *need not* be equal, and if they are not then the map $\Phi_*(\sigma)$ is ill-defined at q.

2. The map ϕ may fail to be surjective. Then, there would be some $q \in N$ such that there is no $X \in \mathrm{im}_{\phi_* \circ \sigma}(M)$ with $\pi(X) = q$ (where $\pi\colon TN \to N$). The map $\Phi_*(\sigma)$ would then be undefined at q.

3. Even if the map ϕ is bijective, its inverse ϕ^{-1} may fail to be smooth. But then $\Phi_*(\sigma)$ would not be guaranteed to be a smooth map.

Of course, everything becomes easier if $\phi\colon M \to N$ is a diffeomorphism.

$$
\begin{array}{ccc}
TM & \xrightarrow{\ \phi_*\ } & TN \\
\sigma \Big\uparrow & & \Big\uparrow \Phi_*(\sigma) \\
M & \xrightarrow{\ \phi\ } & N
\end{array}
$$

If $\sigma \in \Gamma(TM)$, we can define the *push-forward* $\Phi_*(\sigma) \in \Gamma(TN)$ as

$$\Phi_*(\sigma) := \phi_* \circ \sigma \circ \phi^{-1}.$$

More generally, if $\phi\colon M \to N$ is smooth and $\sigma \in \Gamma(TM)$, $\tau \in \Gamma(TN)$, we can define $\Phi_*(\sigma) = \tau$ if σ and τ are *ϕ-related*, i.e. if they satisfy

$$\tau \circ \phi = \phi_* \circ \sigma.$$

We can equip the set $\Gamma(TM)$ with the following operations. The first is our, by now familiar, pointwise addition:

$$
\begin{aligned}
\oplus\colon \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\
(\sigma, \tau) &\mapsto \sigma \oplus \tau,
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma \oplus \tau\colon M &\to \Gamma(TM) \\
p &\mapsto (\sigma \oplus \tau)(p) := \sigma(p) + \tau(p).
\end{aligned}
$$

Note that the $+$ on the right hand side above is the addition in T_pM. More interestingly, we also define the following multiplication operation:

$$
\begin{aligned}
\odot\colon \mathcal{C}^\infty(M) \times \Gamma(TM) &\to \Gamma(TM) \\
(f, \sigma) &\mapsto f \odot \sigma,
\end{aligned}
$$

where

$$
\begin{aligned}
f \odot \sigma\colon M &\to \Gamma(TM) \\
p &\mapsto (f \odot \sigma)(p) := f(p)\sigma(p).
\end{aligned}
$$

Note that since $f \in \mathcal{C}^\infty(M)$, we have $f(p) \in \mathbb{R}$ and hence the multiplication above is the scalar multiplication on T_pM.

If we consider the triple $(\mathcal{C}^\infty(M), +, \bullet)$, where $\bullet$ is pointwise function multiplication as defined in the section on algebras and derivations, then the triple $(\Gamma(TM), \oplus, \odot)$ satisfies

- $(\Gamma(TM), \oplus)$ is an abelian group, with $0 \in \Gamma(TM)$ being the section that maps each $p \in M$ to the zero tangent vector in T_pM;
- $\Gamma(TM) \setminus \{0\}$ satisfies

  i) $\forall\, f \in \mathcal{C}^\infty(M) : \forall\, \sigma, \tau \in \Gamma(TM) \setminus \{0\} : f \odot (\sigma \oplus \tau) = (f \odot \sigma) \oplus (f \odot \tau)$;

  ii) $\forall\, f, g \in \mathcal{C}^\infty(M) : \forall\, \sigma \in \Gamma(TM) \setminus \{0\} : (f + g) \odot \sigma = (f \odot \sigma) \oplus (g \odot \sigma)$;

  iii) $\forall\, f, g \in \mathcal{C}^\infty(M) : \sigma \in \Gamma(TM) \setminus \{0\} : (f \bullet g) \odot \sigma = f \odot (g \odot \sigma)$;

  iv) $\forall\, \sigma \in \Gamma(TM) \setminus \{0\} : 1 \odot \sigma = \sigma$,

  where $1 \in \mathcal{C}^\infty(M)$ maps every $p \in M$ to $1 \in \mathbb{R}$.

These are precisely the axioms for a vector space! The only obstacle to saying that $\Gamma(TM)$ is a vector space over $\mathcal{C}^\infty(M)$ is that the triple $(\mathcal{C}^\infty(M), +, \bullet)$ is *not* an algebraic field, but only a ring. We could simply talk about "vector spaces over rings", but vector spaces over ring have wildly different properties than vector spaces over fields, so much so that they have their own name: *modules*.

Remark .77. Of course, we could have defined $\odot$ simply as pointwise *global* scaling, using the reals $\mathbb{R}$ instead of the real functions $\mathcal{C}^\infty(M)$. Then, since $(\mathbb{R}, +, \cdot)$ is an algebraic field, we would then have the obvious $\mathbb{R}$-vector space structure on $\Gamma(TM)$. However, a basis for this vector space is necessarily uncountably infinite, and hence it does not provide a very useful decomposition for our vector fields.

Instead, the operation $\odot$ that we have defined allows for *local* scaling, i.e. we can scale a vector field by a different value at each point, and a much more useful decomposition of vector fields within the module structure.

C Principal fibre bundles

Very roughly speaking, a principal fibre bundle is a bundle whose typical fibre is a Lie group. Principal fibre bundles are so immensely important because they allow us to understand any fibre bundle with fibre F on which a Lie group G acts. These are then called associated fibre bundles, and will be discussed later on.

C.1 Differential forms

Definition. Let M be a smooth manifold. A *(differential) n-form* on M is a $(0, n)$ smooth tensor field ω which is totally antisymmetric, i.e.

$$\omega(X_1, \dots, X_n) = \operatorname{sgn}(\pi)\, \omega(X_{\pi(1)}, \dots, X_{\pi(n)}),$$

for any $\pi \in S_n$, with $X_i \in \Gamma(TM)$.

Definition. Let $\phi\colon M \to N$ be a smooth map and let $\omega \in \Omega^n(N)$. Then we define the *pull-back* $\Phi^*(\omega) \in \Omega^n(M)$ of ω as

$$\begin{aligned} \Phi^*(\omega)\colon M &\to T^*M \\ p &\mapsto \Phi^*(\omega)(p), \end{aligned}$$

where

$$\Phi^*(\omega)(p)(X_1, \dots, X_n) := \omega(\phi(p))\big(\phi_*(X_1), \dots, \phi_*(X_n)\big),$$

for $X_i \in T_pM$.

The map $\Phi^*\colon \Omega^n(N) \to \Omega^n(M)$ is $\mathbb{R}$-linear, and its action on $\Omega^0(M)$ is simply

$$\begin{aligned} \Phi^*\colon \Omega^0(M) &\to \Omega^0(M) \\ f &\mapsto \Phi^*(f) := f \circ \phi. \end{aligned}$$

Definition. Let M be a smooth manifold. We define the *wedge* (or *exterior*) *product* of forms as the map

$$\begin{aligned} \wedge\colon \Omega^n(M) \times \Omega^m(M) &\to \Omega^{n+m}(M) \\ (\omega, \sigma) &\mapsto \omega \wedge \sigma, \end{aligned}$$

where

$$(\omega \wedge \sigma)(X_1, \dots, X_{n+m}) := \frac{1}{n!\, m!} \sum_{\pi \in S_{n+m}} \operatorname{sgn}(\pi)(\omega \otimes \sigma)(X_{\pi(1)}, \dots, X_{\pi(n+m)})$$

and $X_1, \dots, X_{n+m} \in \Gamma(TM)$. By convention, for any $f, g \in \Omega^0(M)$ and $\omega \in \Omega^n(M)$, we set

$$f \wedge g := fg \qquad \text{and} \qquad f \wedge \omega = \omega \wedge f = f\omega.$$

Example .78. Suppose that $\omega, \sigma \in \Omega^1(M)$. Then, for any $X, Y \in \Gamma(TM)$

$$\begin{aligned}(\omega \wedge \sigma)(X,Y) &= (\omega \otimes \sigma)(X,Y) - (\omega \otimes \sigma)(Y,X)\\ &= (\omega \otimes \sigma)(X,Y) - \omega(Y)\sigma(X)\\ &= (\omega \otimes \sigma)(X,Y) - (\sigma \otimes \omega)(X,Y)\\ &= (\omega \otimes \sigma - \sigma \otimes \omega)(X,Y).\end{aligned}$$

Hence

$$\omega \wedge \sigma = \omega \otimes \sigma - \sigma \otimes \omega.$$

The wedge product is bilinear over $\mathcal{C}^\infty(M)$, that is

$$(f\omega_1 + \omega_2) \wedge \sigma = f\,\omega_1 \wedge \sigma + \omega_2 \wedge \sigma,$$

for all $f \in \mathcal{C}^\infty(M)$, $\omega_1, \omega_2 \in \Omega^n(M)$ and $\sigma \in \Omega^m(M)$, and similarly for the second argument.

Remark .79. If (U, x) is a chart on M, then every n-form $\omega \in \Omega^n(U)$ can be expressed locally on U as

$$\omega = \omega_{a_1 \cdots a_n} \, \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n},$$

where $\omega_{a_1 \cdots a_n} \in \mathcal{C}^\infty(U)$ and $1 \leq a_1 < \cdots < a_n \leq \dim M$. The $\mathrm{d}x^{a_i}$ appearing above are the covector fields (1-forms)

$$\mathrm{d}x^{a_i} : p \mapsto \mathrm{d}_p x^{a_i}.$$

The pull-back distributes over the wedge product.

Theorem .80. *Let $\phi \colon M \to N$ be smooth, $\omega \in \Omega^n(N)$ and $\sigma \in \Omega^m(N)$. Then, we have*

$$\Phi^*(\omega \wedge \sigma) = \Phi^*(\omega) \wedge \Phi^*(\sigma).$$

Proof. Let $p \in M$ and $X_1, \ldots, X_{n+m} \in T_pM$. Then we have

$$\begin{aligned}&\big(\Phi^*(\omega) \wedge \Phi^*(\sigma)\big)(p)(X_1, \ldots, X_{n+m})\\ &\quad = \frac{1}{n!\,m!} \sum_{\pi \in S_{n+m}} \operatorname{sgn}(\pi) \big(\Phi^*(\omega) \otimes \Phi^*(\sigma)\big)(p)(X_{\pi(1)}, \ldots, X_{\pi(n+m)})\\ &\quad = \frac{1}{n!\,m!} \sum_{\pi \in S_{n+m}} \operatorname{sgn}(\pi) \Phi^*(\omega)(p)(X_{\pi(1)}, \ldots, X_{\pi(n)})\\ &\qquad\qquad\qquad\qquad \Phi^*(\sigma)(p)(X_{\pi(n+1)}, \ldots, X_{\pi(n+m)})\\ &\quad = \frac{1}{n!\,m!} \sum_{\pi \in S_{n+m}} \operatorname{sgn}(\pi) \omega(\phi(p))\big(\phi_*(X_{\pi(1)}), \ldots, \phi_*(X_{\pi(n)})\big)\\ &\qquad\qquad\qquad\qquad \sigma(\phi(p))\big(\phi_*(X_{\pi(n+1)}), \ldots, \phi_*(X_{\pi(n+m)})\big)\\ &\quad = \frac{1}{n!\,m!} \sum_{\pi \in S_{n+m}} \operatorname{sgn}(\pi) (\omega \otimes \sigma)(\phi(p))\big(\phi_*(X_{\pi(1)}), \ldots, \phi_*(X_{\pi(n+m)})\big)\\ &\quad = (\omega \wedge \sigma)(\phi(p))\big(\phi_*(X_1), \ldots, \phi_*(X_{n+m})\big)\\ &\quad = \Phi^*(\omega \wedge \sigma)(p)(X_1, \ldots, X_{n+m}).\end{aligned}$$

Since $p \in M$ was arbitrary, the statement follows. □

C.2 The exterior derivative

We will need the following definition.

Definition. Let M be a smooth manifold and let $X, Y \in \Gamma(TM)$. The commutator (or Lie bracket) of X and Y is defined as

$$\begin{aligned}[X,Y] \colon \mathcal{C}^\infty(M) &\xrightarrow{\sim} \mathcal{C}^\infty(M)\\ f &\mapsto [X,Y](f) := X(Y(f)) - Y(X(f)),\end{aligned}$$

where we are using the definition of vector fields as $\mathbb{R}$-linear maps $\mathcal{C}^\infty(M) \xrightarrow{\sim} \mathcal{C}^\infty(M)$.

Definition. The *exterior derivative* on M is the $\mathbb{R}$-linear operator

$$\begin{aligned}\mathrm{d} \colon \Omega^n(M) &\xrightarrow{\sim} \Omega^{n+1}(M)\\ \omega &\mapsto \mathrm{d}\omega\end{aligned}$$

with $\mathrm{d}\omega$ being defined as

$$\begin{aligned}\mathrm{d}\omega(X_1,\ldots,X_{n+1}) := &\sum_{i=1}^{n+1}(-1)^{i+1}\, X_i\big(\omega(X_1,\ldots,\widehat{X_i},\ldots,X_{n+1})\big)\\ &+\sum_{i<j}(-1)^{i+j}\,\omega\big([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_{n+1}\big),\end{aligned}$$

where $X_i \in \Gamma(TM)$ and the hat denotes omissions.

Example .81. In the case $n = 1$, the form $\mathrm{d}\omega \in \Omega^2(M)$ is given by

$$\mathrm{d}\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

Let us check that this is indeed a 2-form, i.e. an antisymmetric, $\mathcal{C}^\infty(M)$-multilinear map

$$\mathrm{d}\omega \colon \Gamma(TM) \times \Gamma(TM) \to \mathcal{C}^\infty(M).$$

By using the antisymmetry of the Lie bracket, we immediately get

$$\mathrm{d}\omega(X,Y) = -\,\mathrm{d}\omega(Y,X).$$

Moreover, thanks to this identity, it suffices to check $\mathcal{C}^\infty(M)$-linearity in the first argument only. Additivity is easily checked

$$\begin{aligned}\mathrm{d}\omega(X_1+X_2,Y) &= (X_1+X_2)(\omega(Y)) - Y(\omega(X_1+X_2)) - \omega([X_1+X_2,Y])\\ &= X_1(\omega(Y)) + X_2(\omega(Y)) - Y(\omega(X_1)+\omega(X_2)) - \omega([X_1,Y]+[X_2,Y])\\ &= X_1(\omega(Y)) + X_2(\omega(Y)) - Y(\omega(X_1)) - Y(\omega(X_2)) - \omega([X_1,Y]) - \omega([X_2,Y])\\ &= \mathrm{d}\omega(X_1,Y) + \mathrm{d}\omega(X_2,Y).\end{aligned}$$

For $\mathcal{C}^\infty(M)$-scaling, first we calculate $[fX, Y]$. Let $g \in \mathcal{C}^\infty(M)$. Then

$$\begin{aligned}[fX, Y](g) &= fX(Y(g)) - Y(fX(g)) \\ &= fX(Y(g)) - fY(X(g)) - Y(f)X(g) \\ &= f(X(Y(g)) - Y(X(g))) - Y(f)X(g) \\ &= f[X, Y](g) - Y(f)X(g) \\ &= (f[X, Y] - Y(f)X)(g).\end{aligned}$$

Therefore

$$[fX, Y] = f[X, Y] - Y(f)X.$$

Hence, we can calculate

$$\begin{aligned}d\omega(fX, Y) &= fX(\omega(Y)) - Y(\omega(fX)) - \omega([fX, Y]) \\ &= fX(\omega(Y)) - Y(f\omega(X)) - \omega(f[X, Y] - Y(f)X) \\ &= fX(\omega(Y)) - fY(\omega(X)) - Y(f)\omega(X) - f\omega([X, Y]) + \omega(Y(f)X) \\ &= fX(\omega(Y)) - fY(\omega(X)) - \cancel{Y(f)\omega(X)} - f\omega([X, Y]) + \cancel{Y(f)\omega(X)} \\ &= f\, d\omega(X, Y),\end{aligned}$$

which is what we wanted.

The exterior derivative satisfies a graded version of the Leibniz rule with respect to the wedge product.

Theorem .82. *Let $\omega \in \Omega^n(M)$ and $\sigma \in \Omega^m(M)$. Then*

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^n\, \omega \wedge d\sigma.$$

Proof. We will work in local coordinates. Let (U, x) be a chart on M and write

$$\begin{aligned}\omega &= \omega_{a_1 \cdots a_n} dx^{a_1} \wedge \cdots \wedge dx^{a_n} =: \omega_A dx^A \\ \sigma &= \sigma_{b_1 \cdots b_m} dx^{b_1} \wedge \cdots \wedge dx^{b_m} =: \sigma_B dx^B.\end{aligned}$$

Locally, the exterior derivative operator d acts as

$$d\omega = d\omega_A \wedge dx^A.$$

Hence

$$\begin{aligned}d(\omega \wedge \sigma) &= d(\omega_A \sigma_B\, dx^A \wedge dx^B) \\ &= d(\omega_A \sigma_B) \wedge dx^A \wedge dx^B \\ &= (\sigma_B d\omega_A + \omega_A d\sigma_B) \wedge dx^A \wedge dx^B \\ &= \sigma_B d\omega_A \wedge dx^A \wedge dx^B + \omega_A d\sigma_B \wedge dx^A \wedge dx^B \\ &= \sigma_B d\omega_A \wedge dx^A \wedge dx^B + (-1)^n \omega_A dx^A \wedge d\sigma_B \wedge dx^B \\ &= \sigma_B d\omega \wedge dx^B + (-1)^n \omega_A dx^A \wedge d\sigma \\ &= d\omega \wedge \sigma + (-1)^n\, \omega \wedge d\sigma\end{aligned}$$

since we have "anticommuted" the 1-form $d\sigma_B$ through the n-form dx^A, picking up n minus signs in the process. □

An important property of the exterior derivative is the following.

Theorem .83. *Let $\phi\colon M \to N$ be smooth. For any $\omega \in \Omega^n(N)$, we have*

$$\Phi^*(\mathrm{d}\omega) = \mathrm{d}(\Phi^*(\omega)).$$

Proof (sketch). We first show that this holds for 0-forms (i.e. smooth functions).

Let $f \in \mathcal{C}^\infty(N)$, $p \in M$ and $X \in T_pM$. Then

$$\begin{aligned}
\Phi^*(\mathrm{d}f)(p)(X) &= \mathrm{d}f(\phi(p))(\phi_*(X)) && \text{(definition of } \Phi^*) \\
&= \phi_*(X)(f) && \text{(definition of } \mathrm{d}f) \\
&= X(f \circ \phi) && \text{(definition of } \phi_*) \\
&= \mathrm{d}(f \circ \phi)(p)(X) && \text{(definition of } \mathrm{d}(f \circ \phi)) \\
&= \mathrm{d}(\Phi^*(f))(p)(X) && \text{(definition of } \Phi^*),
\end{aligned}$$

so that we have $\Phi^*(\mathrm{d}f) = \mathrm{d}(\Phi^*(f))$.

The general result follows from the linearity of Φ^* and the fact that the pull-back distributes over the wedge product. □

Definition. Let M be a smooth manifold. A 2-form $\omega \in \Omega^2(M)$ is said to be a *symplectic form* on M if $\mathrm{d}\omega = 0$ and if it is non-degenerate, i.e.

$$(\forall\, Y \in \Gamma(TM) : \omega(X, Y) = 0) \Rightarrow X = 0.$$

A manifold equipped with a symplectic form is called a *symplectic manifold.*

C.3 Lie group actions on a manifold

Definition. Let $(G, \bullet)$ be a Lie group and let M be a smooth manifold. A smooth map

$$\begin{aligned}
\triangleright\colon G \times M &\to M \\
(g, p) &\mapsto g \triangleright p
\end{aligned}$$

satisfying

i) $\forall\, p \in M : \; e \triangleright p = p$;

ii) $\forall\, g_1, g_2 \in G : \forall\, p \in M : \; (g_1 \bullet g_2) \triangleright p = g_1 \triangleright (g_2 \triangleright p)$,

is called a *left Lie group action*, or *left G-action*, on M.

Definition. A manifold equipped with a left G-action is called a *left G-manifold.*

Definition. Similarly, a *right G-action* on M is a smooth map

$$\begin{aligned}
\triangleleft\colon M \times G &\to M \\
(p, g) &\mapsto p \triangleleft g
\end{aligned}$$

satisfying

i) $\forall\, p \in M: \ p \triangleleft g = p$;

ii) $\forall\, g_1, g_2 \in G : \forall\, p \in M: \ p \triangleleft (g_1 \bullet g_2) = (p \triangleleft g_1) \triangleleft g_2$.

Proposition .84. *Let $\triangleright$ be a left G-action on M. Then*

$$\begin{aligned} \triangleleft \colon M \times G &\to M \\ (p, g) &\mapsto p \triangleleft g := g^{-1} \triangleright p \end{aligned}$$

is a right G-action on M.

Definition. Let G, H be Lie groups, let $\rho \colon G \to H$ be a Lie group homomorphism and let

$$\begin{aligned} \triangleright \colon G \times M \to M, \\ \blacktriangleright \colon H \times N \to N \end{aligned}$$

be left actions of G and H on some smooth manifolds M and N, respectively. Then, a smooth map $f \colon M \to N$ is said to be *ρ-equivariant* if the diagram

$$\begin{array}{ccc} G \times M & \xrightarrow{\ \rho \times f\ } & H \times N \\ \Big\downarrow{\scriptstyle \triangleright} & & \Big\downarrow{\scriptstyle \blacktriangleright} \\ M & \xrightarrow{\quad f \quad} & N \end{array}$$

where $(\rho \times f)(g, p) := (\rho(g), f(p)) \in H \times N$, commutes. Equivalently,

$$\forall\, g \in G : \forall\, p \in M: \ f(g \triangleright p) = \rho(g) \blacktriangleright f(p).$$

In other words, if $\rho \colon G \to H$ is a Lie group homomorphism, then the ρ-equivariant maps are the "action-preserving" maps between the G-manifold M and the H-manifold N.

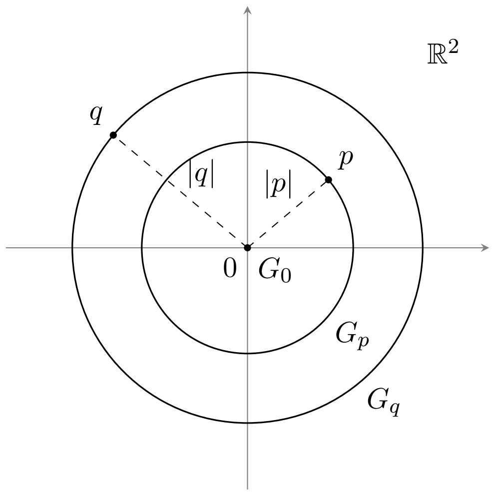
Remark .85. Note that by setting $\rho = \mathrm{id}_G$ or $f = \mathrm{id}_M$, the notion of f being ρ-equivariant reduces to what we might call a homomorphism of G-manifolds in the former case, and a homomorphism of left actions on M in the latter.

Definition. Let $\triangleright \colon G \times M \to M$ be a left G-action. For each $p \in M$, we define the *orbit* of p as the set

$$G_p := \{q \in M \mid \exists\, g \in G : q = g \triangleright p\}.$$

Alternatively, the orbit of p is the image of G under the map $(- \triangleright p)$. It consists of all the points in M that can be reached from p by successive applications of the action $\triangleright$.

Example .86. Consider the action induced by representation of $\mathrm{SO}(2, \mathbb{R})$ as rotation matrices in $\mathrm{End}(\mathbb{R}^2)$. The orbit of any $p \in \mathbb{R}^2$ is the circle of radius $|p|$ centred at the origin.



It should be intuitively clear from the definition that the orbits of two points are either disjoint or coincide. In fact, we have the following.

Proposition .87. *Let $\triangleright\colon G \times M \to M$ be an action on M. Define a relation on M*

$$p \sim q \;:\Leftrightarrow\; \exists\, g \in G : q = g \triangleright p.$$

Then $\sim$ is an equivalence relation on M.

The equivalence classes of $\sim$ are, by definition, the orbits.

Definition. Let $\triangleright\colon G \times M \to M$ be an action on M. The *orbit space* of M is

$$M/G := M/\!\sim\, = \{G_p \mid p \in M\}.$$

Example .88. The orbit space of our previous $\mathrm{SO}(2,\mathbb{R})$-action on $\mathbb{R}^2$ is the partition of $\mathbb{R}^2$ into concentric circles centred at the origin, plus the origin itself.

Definition. Let $\triangleright\colon G \times M \to M$ be a G-action on M. The *stabiliser* of $p \in M$ is

$$S_p := \{g \in G \mid g \triangleright p = p\}.$$

Note that for each $p \in M$, the stabiliser S_p is a subgroup of G.

Example .89. In our $\mathrm{SO}(2,\mathbb{R})$ example, we have $S_p = \{\mathrm{id}_{\mathbb{R}^2}\}$ for $p \neq 0$ and $S_0 = \mathrm{SO}(2,\mathbb{R})$.

Definition. A left G-action $\triangleright\colon G \times M \to M$ is said to be

i) *free* if for all $p \in M$, we have $S_p = \{e\}$;

ii) *transitive* if for all $p, q \in M$, there exists $g \in G$ such that $p = g \triangleright p$.

Example .90. The action $\triangleright\colon G \times V \to V$ induced by a representation $R\colon G \to \mathrm{GL}(V)$ is never free since we always have $S_0 = G$.

Example .91. Consider the action $\triangleright\colon \mathrm{T}(n) \times \mathbb{R}^n \to \mathbb{R}^n$ of the n-dimensional translation group $\mathrm{T}(n)$ on $\mathbb{R}^n$. We have, rather trivially, $\mathrm{T}(n)_p = \mathbb{R}^n$ for every $p \in \mathbb{R}^n$.It is also easy to show that this action is free and transitive.

Proposition .92. *Let* $\triangleright\colon G \times M \to M$ *be a free action. Then*

$$g_1 \triangleright p = g_2 \triangleright p \quad \Leftrightarrow \quad g_1 = g_2.$$

Proposition .93. *If* $\triangleright\colon G \times M \to M$ *is a free action, then*

$$\forall\, p \in G : \; G_p \cong_{\text{diff}} G.$$

Example .94. Define $\triangleright\colon \operatorname{SO}(2,\mathbb{R}) \times \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ to coincide with the action induced by the representation of $\operatorname{SO}(2,\mathbb{R}^2)$ on $\mathbb{R}^2$ for each non-zero point of $\mathbb{R}^2$. Then this action is free, since we have $S_p = \{\text{id}_{\mathbb{R}^2}\}$ for $p \neq 0$, and the previous proposition implies

$$\forall\, p \in \mathbb{R}^2 \setminus \{0\} : \; \operatorname{SO}(2,\mathbb{R})_p \cong_{\text{diff}} \operatorname{SO}(2,\mathbb{R}) \cong_{\text{diff}} S^1.$$

C.4 Principal fibre bundles

Definition. Let G be a Lie group. A smooth bundle (E, π, M) is called a *principal G-bundle* if E is equipped with a free right G-action and

$$\begin{array}{ccc} E & & E \\ {\scriptstyle\pi}\big\downarrow & \cong_{\text{bdl}} & \big\downarrow{\scriptstyle\rho} \\ M & & E/G \end{array}$$

where ρ is the quotient map, defined by sending each $p \in E$ to its equivalence class (i.e. orbit) in the orbit space E/G.

Observe that since the right action of G on E is free, for each $p \in E$ we have

$$\operatorname{preim}_\rho(G_p) = G_p \cong_{\text{diff}} G.$$

Remark .95. A slight generalisation would be to consider smooth bundles $E \xrightarrow{\pi} M$ where E is equipped with a right G-action which is free and transitive on each fibre of $E \xrightarrow{\pi} M$. The isomorphism in our definition enforces the fibre-wise transitivity since G acts transitively on each G_p by the definition of orbit.

C.5 Principal bundle morphisms

Recall that a bundle morphism (also called simply a bundle map) between two bundles (E, π, M) and (E', π', M') is a pair of maps (u, f) such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\quad u \quad} & E' \\ {\scriptstyle\pi}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{\quad f \quad} & M' \end{array}$$

commutes, that is, $f \circ \pi = \pi' \circ u$.

Definition. Let (P, π, M) and (Q, π', N) be principal G-bundles. A *principal bundle morphism* from (P, π, M) to (Q, π', N) is a pair of smooth maps (u, f) such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\;u\;} & Q \\ {\scriptstyle \triangleleft G}\uparrow & & \uparrow{\scriptstyle \blacktriangleleft G} \\ P & \xrightarrow{\;u\;} & Q \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{\;f\;} & N \end{array}$$

commutes, that is for all $p \in P$ and $g \in G$, we have

$$\begin{aligned} (f \circ \pi)(p) &= (\pi' \circ u)(p) \\ u(p \triangleleft g) &= u(p) \blacktriangleleft g. \end{aligned}$$

Note that $P \xrightarrow{\triangleleft G} P$ is a shorthand for the inclusion of P into the product $P \times G$ followed by the right action $\triangleleft$, i.e.

$$P \xrightarrow{\triangleleft G} P \qquad = \qquad P \xrightarrow{\;i_1\;} P \times G \xrightarrow{\;\triangleleft\;} P$$

and similarly for $Q \xrightarrow{\blacktriangleleft G} Q$.

Definition. A principal bundle morphism between two principal G-bundles is an *isomorphism* (or *diffeomorphism*) *of principal bundles* if it is also a bundle isomorphism.

Remark .96. Note that the passage from principal bundle morphism to principal bundle isomorphism does not require any extra condition involving the Lie group G. We will soon see that this is because the two bundles are both principal G-bundles. We can further generalise the notion of principal bundle morphism as follows.

Definition. Let (P, π, M) be a principal G-bundle, let (Q, π', N) be a principal H-bundle, and let $\rho \colon G \to H$ be a Lie group homomorphism. A *principal bundle morphism* from (P, π, M) to (Q, π', N) is a pair of smooth maps (u, f) such that the diagram

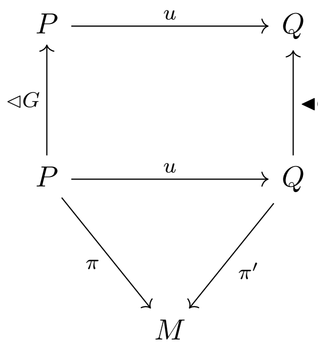
$$\begin{array}{ccc} P & \xrightarrow{\;u\;} & Q \\ {\scriptstyle \triangleleft}\uparrow & & \uparrow{\scriptstyle \blacktriangleleft} \\ P \times G & \xrightarrow{\;u \times \rho\;} & Q \times H \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_1} \\ P & \xrightarrow{\;u\;} & Q \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{\;f\;} & N \end{array}$$

commutes, that is $f \circ \pi = \pi' \circ u$ and u is a ρ-equivariant map

$$\forall\, p \in P : \forall\, g \in G : \ u(p \triangleleft g) = u(p) \blacktriangleleft \rho(g).$$

Definition. A principal bundle morphism between principal G-bundle and a principal H-bundle is an *isomorphism* (or *diffeomorphism*) *of principal bundles* if it is also a bundle isomorphism and ρ is a Lie group isomorphism.

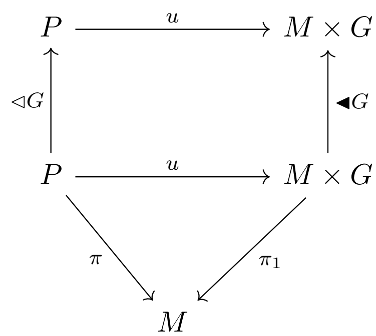
Lemma .97. *Let (P, π, M) and (Q, π', M) be principal G-bundles over the same base manifold M. Then, any $u \colon P \to Q$ such that (u, id_M) is a principal bundle morphism is necessarily a diffeomorphism.*



Definition. A principal G-bundle (P, π, M) if it is called *trivial* if it is isomorphic as a principal G-bundle to the principal G-bundle $(M \times G, \pi_1, M)$ where π_1 is the projection onto the first component and the action is defined as

$$\begin{aligned} \blacktriangleleft \colon (M \times G) \times G &\to M \times G \\ ((p, g), g') &\mapsto (p, g) \blacktriangleleft g' := (p, g \bullet g'). \end{aligned}$$

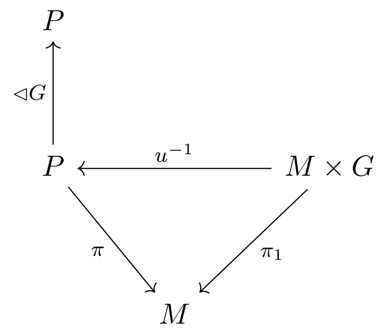
By the previous lemma, a principal G-bundle (P, π, M) is trivial if there exists a smooth map $u \colon P \to M \times G$ such that the following diagram commutes.



The following result provides a necessary and sufficient criterion for when a principal bundle is trivial. Note that while we have used the lower case letter p almost exclusively to denote points of the base manifold M, in the next proof we will use it to denote points of the total space P instead.

Theorem .98. *A principal G-bundle (P, π, M) is trivial if, and only if, there exists a smooth section $\sigma \in \Gamma(P)$, that is, a smooth $\sigma \colon M \to P$ such that $\pi \circ \sigma = \mathrm{id}_M$.*

Proof. (⇒) Suppose that (P, π, M) is trivial. Then there exists a diffeomorphism $u\colon P \to M \times G$ which make the following diagram commute



We can define

$$\begin{aligned} \sigma\colon M &\to P \\ m &\mapsto u^{-1}(m, e), \end{aligned}$$

where e is the identity of G. Then σ is smooth since it is the composition of u^{-1} with the map $p \mapsto (p, e)$, which are both smooth. We also have

$$(\pi \circ \sigma)(pm = \pi(u^{-1}(m, e)) = \pi_1(m, e) = m,$$

for all $m \in M$, hence $\pi \circ \sigma = \mathrm{id}_M$ and thus $\sigma \in \Gamma(P)$.

(⇐) Suppose that there exists a smooth section $\sigma\colon M \to P$. Let $p \in P$ and consider the point $\sigma(\pi(p)) \in P$. We have

$$\pi(\sigma(\pi(p))) = \mathrm{id}_M(\pi(p)) = \pi(p),$$

hence $\sigma(\pi(p))$ and p belong to the same fibre, and thus there exists a unique group element in G which links the two points via $\triangleleft$. Since this element depends on both σ and p, let us denote it by $\chi_\sigma(p)$. Then, χ_σ defines a function

$$\begin{aligned} \chi_\sigma\colon P &\to G \\ p &\mapsto \chi_\sigma(p) \end{aligned}$$

and we can write

$$\forall\, p \in P:\ p = \sigma(\pi(p)) \triangleleft \chi_\sigma(p).$$

In particular, for any other $g \in G$ we have $p \triangleleft g \in P$ and thus

$$p \triangleleft g = \sigma(\pi(p \triangleleft g)) \triangleleft \chi_\sigma(p \triangleleft g) = \sigma(\pi(p)) \triangleleft \chi_\sigma(p \triangleleft g),$$

where the second equality follows from the fact that the fibres of P are precisely the orbits under the action of G.

On the other hand, we can act on the right with an arbitrary $g \in G$ directly to obtain

$$p \triangleleft g = (\sigma(\pi(p)) \triangleleft \chi_\sigma(p)) \triangleleft g = \sigma(\pi(p)) \triangleleft (\chi_\sigma(p) \bullet g).$$

Combining the last two equations yields

$$\sigma(\pi(p)) \triangleleft \chi_\sigma(p \triangleleft g) = \sigma(\pi(p)) \triangleleft (\chi_\sigma(p) \bullet g)$$

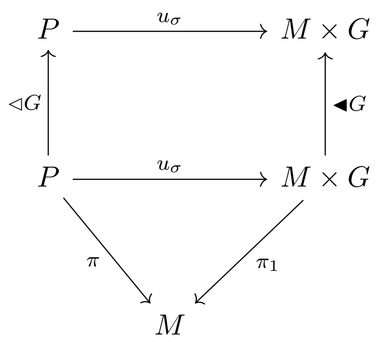
and hence

$$\chi_\sigma(p \triangleleft g) = (\chi_\sigma(p) \bullet g).$$

We can now define the map

$$\begin{aligned} u_\sigma \colon P &\to M \times G \\ p &\mapsto (\pi(p), \chi_\sigma(p)). \end{aligned}$$

By our previous lemma, it suffices to show that u_σ is a principal bundle map.



By definition, we have

$$(\pi_1 \circ u_\sigma)(p) = \pi_1(\pi(p), \chi_\sigma(p)) = \pi(p)$$

for all $p \in P$, so the lower triangle commutes. Moreover, we have

$$\begin{aligned} u_\sigma(p \triangleleft g) &= (\pi(p \triangleleft g), \chi_\sigma(p \triangleleft g)) \\ &= (\pi(p), \chi_\sigma(p) \bullet g)) \\ &= (\pi(p), \chi_\sigma(p)) \blacktriangleleft g \\ &= u_\sigma(p) \blacktriangleleft g \end{aligned}$$

for all $p \in P$ and $g \in G$, so the upper square also commutes and hence (P, π, M) is a trivial bundle. □

D Associated fibre bundles

An associated fibre bundle is a fibre bundle which is associated (in a precise sense) to a principal G-bundle.

D.1 Associated fibre bundles

Definition. Let (P, π, M) be a principal G-bundle and let F be a smooth manifold equipped with a left G-action $\triangleright$. We define

i) $P_F := (P \times F)/{\sim_G}$, where $\sim_G$ is the equivalence relation

$$(p, f) \sim_G (p', f') \quad :\Leftrightarrow \quad \exists\, g \in G : \begin{cases} p' = p \triangleleft g \\ f' = g^{-1} \triangleright f \end{cases}$$

We denote the points of P_F as $[p, f]$.

ii) The map

$$\begin{aligned} \pi_F \colon \quad P_F &\to M \\ [p, f] &\mapsto \pi(p), \end{aligned}$$

which is well-defined since, if $[p', f'] = [p, f]$, then for some $g \in G$

$$\pi_F([p', f']) = \pi_F([p \triangleleft g, g^{-1} \triangleright f]) := \pi(p \triangleleft g) = \pi(p) =: \pi_F([p, f]).$$

The *associated bundle* (to (P, π, M), F and $\triangleright$) is the bundle (P_F, π_F, M).

Definition. Let M be a smooth manifold and let (LM, π, M) be its frame bundle, with right $\mathrm{GL}(d, \mathbb{R})$-action as above. Let $F := (\mathbb{R}^d)^{\times p} \times (\mathbb{R}^{d*})^{\times q}$ and define a left $\mathrm{GL}(d, \mathbb{R})$-action on F by

$$(g \triangleright f)^{a_1 \cdots a_p}{}_{b_1 \cdots b_q} := (\det g^{-1})^{\omega}\, g^{a_1}{}_{\widetilde{a}_1} \cdots g^{a_p}{}_{\widetilde{a}_p} (g^{-1})^{\widetilde{b}_1}{}_{b_1} \cdots (g^{-1})^{\widetilde{b}_q}{}_{b_q}\, f^{\widetilde{a}_1 \cdots \widetilde{a}_p}{}_{\widetilde{b}_1 \cdots \widetilde{b}_q},$$

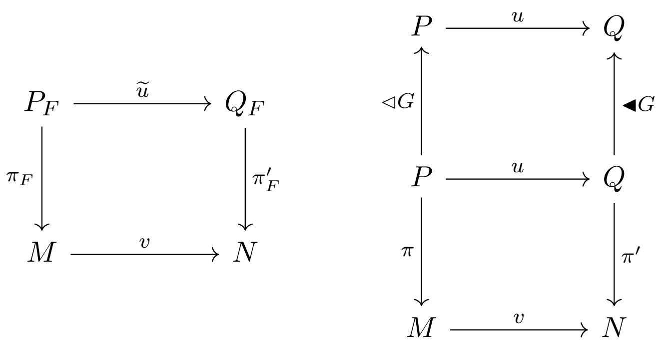
where $\omega \in \mathbb{Z}$. Then the associated bundle (LM_F, π_F, M) is called the *(p, q)-tensor ω-density bundle* on M, and its sections are called *(p, q)-tensor densities of weight ω*.

D.2 Associated bundle morphisms

Definition. Let (P_F, π_F, M) to (Q_F, π'_F, N) be the associated bundles (with the same fibre F) of two principal G-bundles (P, π, M) and (Q, π', N). An *associated bundle map* between the associated bundles is a bundle map $(\widetilde{u}, v)$ between them such that for some u, the pair (u, v) is a principal bundle map between the underlying principal G-bundles and

$$\widetilde{u}([p, f]) := [u(p), f].$$

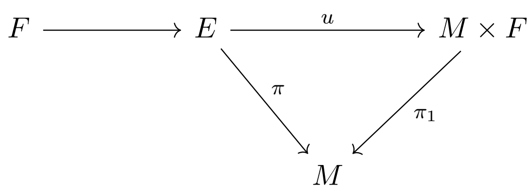
Equivalently, the following diagrams both commute.



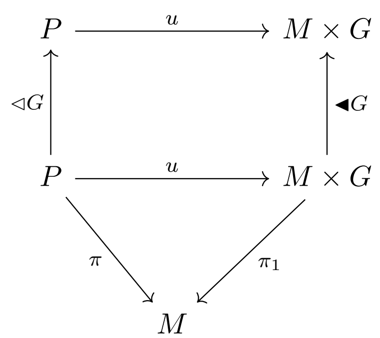
Definition. An associated bundle map $(\widetilde{u}, v)$ is an *associated bundle isomorphism* if $\widetilde{u}$ and v are invertible and $(\widetilde{u}^{-1}, v^{-1})$ is also an associated bundle map.

Remark .99. Note that two associated F-fibre bundles may be isomorphic as bundles but not as associated bundles. In other words, there may exist a bundle isomorphism between them, but there may not exist any bundle isomorphism between them which can be written as in the definition for some principal bundle isomorphism between the underlying principal bundles.

Recall that an F-fibre bundle (E, π, M) is called trivial if there exists a bundle isomorphism



while a principal G-bundle is called trivial if there exists a principal bundle isomorphism



Definition. An associated bundle (P_F, π_F, M) is called *trivial* if the underlying principal G-bundle (P, π, M) is trivial.

Proposition .100. *A trivial associated bundle is a trivial fibre bundle.*

Note that the converse does not hold. An associated bundle can be trivial as a fibre bundle but not as an associated bundle, i.e. the underlying principal fibre bundle need not be trivial simply because the associated bundle is trivial as a fibre bundle.

Definition. Let H be a closed Lie subgroup of G. Let (P, π, M) be a principal H-bundle and (Q, π', M) a principal G-bundle. If there exists a principal bundle map from (P, π, M) to (Q, π', M), i.e. a smooth bundle map which is equivariant with respect to the inclusion of H into G, then (P, π, M) is called an *H-restriction* of (Q, π', M), while (Q, π', M) is called a *G-extension* of (P, π, M).

Theorem .101. *Let H be a closed Lie subgroup of G.*

i) Any principal H-bundle can be extended to a principal G-bundle.

ii) A principal G-bundle (P, π, M) can be restricted to a principal H-bundle if, and only if, the bundle $(P/H, \pi', M)$ has a section.

Example .102. i) The bundle $(LM/\operatorname{SO}(d), \pi, M)$ always has a section, and since $\operatorname{SO}(d)$ is a closed Lie subgroup of $\operatorname{GL}(d, \mathbb{R})$, the frame bundle can be restricted to a principal $\operatorname{SO}(d)$-bundle. This is related to the fact that any manifold can be equipped with a Riemannian metric.

ii) The bundle $(LM/\operatorname{SO}(1, d-1), \pi, M)$ may or may not have a section. For example, the bundle $(LS^2/\operatorname{SO}(1,1), \pi, S^2)$ does not admit any section, and hence we cannot restrict $(LS^2/\operatorname{SO}(1,1), \pi, S^2)$ to a principal $\operatorname{SO}(1,1)$-bundle, even though $\operatorname{SO}(1,1)$ is a closed Lie subgroup of $\operatorname{GL}(2, \mathbb{R})$. This is related to the fact that the 2-sphere cannot be equipped with a Lorentzian metric.

E Connections and connection 1-forms

In elementary courses on differential geometry or general relativity, the notions of connection, parallel transport and covariant derivative are often confused with one another. Sometimes, the terms are even used as synonyms. If you have seen any of that before, it is probably best to forget about it for the time being. The conceptual sequence "connection, parallel transport covariant derivative" is in decreasing order of generality, and it should be clear that treating these terms as synonyms will inevitably lead to confusion. We will now discuss the first of these in some detail.

E.1 Connections on a principal bundle

Definition. Let (P, π, M) be a principal bundle and let $p \in P$. The *vertical subspace* at p is the vector subspace of T_pP given by

$$\begin{aligned} V_pP &:= \ker((\pi_*)_p) \\ &= \{X_p \in T_pP \mid (\pi_*)_p(X_p) = 0\}. \end{aligned}$$

Lemma .103. *For all $A \in T_eG$ and $p \in P$, we have $X_p^A \in V_pP$.*

Proof. Since the action of G simply permutes the elements within each fibre, we have

$$\pi(p) = \pi(p \triangleleft \exp(tA)),$$

for any t. Let $f \in \mathcal{C}^\infty(M)$ be arbitrary. Then

$$\begin{aligned} (\pi_*)_p X_p^A(f) &= X_p^A(f \circ \pi) \\ &= [(f \circ \pi)(p \triangleleft \exp(tA))]'(0) \\ &= [f(\pi(p))]'(0) \\ &= 0, \end{aligned}$$

since $f(\pi(p))$ is constant. Hence $X_p^A \in V_pP$. Alternatively, one can also argue that $(\pi_*)_p X_p^A$ is the tangent vector to a constant curve on M. □

In particular, the map $i_p \colon T_eG \xrightarrow{\sim} V_pP$ is now a bijection.

Definition. Let (P, π, M) be a principal bundle and let $p \in P$. A *horizontal subspace* at p a vector subspace H_pP of T_pP which is complementary to V_pP, i.e.

$$T_pP = H_pP \oplus V_pP.$$

The choice of horizontal space at $p \in P$ is not unique. However, once a choice is made, there is a unique decomposition of each $X_p \in T_pP$ as

$$X_p = \mathrm{hor}(X_p) + \mathrm{ver}(X_p),$$

with $\mathrm{hor}(X_p) \in H_pP$ and $\mathrm{ver}(X_p) \in V_pP$.

Definition. A *connection* on a principal G-bundle (P, π, M) is a choice of horizontal space at each $p \in P$ such that

i) For all $g \in G$, $p \in P$ and $X_p \in H_pP$, we have

$$(\triangleleft\, g)_* X_p \in H_{p \triangleleft g} P,$$

where $(\triangleleft\, g)_*$ is the push-forward of the map $(- \triangleleft g) \colon P \to P$ and it is a bijection. We can also write this condition more concisely as

$$(\triangleleft\, g)_*(H_pP) = H_{p \triangleleft g} P.$$

ii) For every smooth $X \in \Gamma(TP)$, the two summands in the unique decomposition

$$X|_p = \operatorname{hor}(X|_p) + \operatorname{ver}(X|_p)$$

at each $p \in P$, extend to smooth $\operatorname{hor}(X), \operatorname{ver}(X) \in \Gamma(TM)$.

E.2 Connection one-forms

Definition. The map $\omega \colon p \to \omega_p$ sending each $p \in P$ to the T_eG-valued one-form ω_p is called the *connection one-form* with respect to the connection.

Lemma .104. *For all $p \in P$, $g \in G$ and $A \in T_eG$, we have*

$$(\triangleleft\, g)_* X_p^A = X_{p \triangleleft g}^{(\mathrm{Ad}_{g^{-1}})_* A}.$$

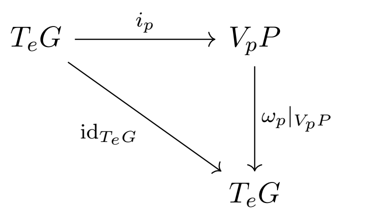
Proof. Let $f \in \mathcal{C}^\infty(P)$ be arbitrary. We have

$$\begin{aligned}
(\triangleleft\, g)_* X_p^A(f) &= X_p^A(f \circ (- \triangleleft g)) \\
&= [f(p \triangleleft \exp(tA) \triangleleft g)]'(0) \\
&= [f(p \triangleleft g \triangleleft g^{-1} \triangleleft \exp(tA) \triangleleft g)]'(0) \\
&= [f(p \triangleleft g \triangleleft (g^{-1} \bullet \exp(tA) \bullet g)]'(0) \\
&= [f(p \triangleleft g \triangleleft \mathrm{Ad}_{g^{-1}}(\exp(tA))]'(0) \\
&= [f(p \triangleleft g \triangleleft \exp(t(\mathrm{Ad}_{g^{-1}})_* A)]'(0) \\
&= X_{p \triangleleft g}^{(\mathrm{Ad}_{g^{-1}})_* A}(f),
\end{aligned}$$

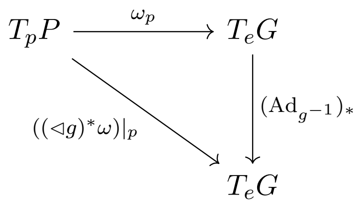
which is what we wanted. □

Theorem .105. *A connection one-form ω with respect to a connection satisfies*

a) For all $p \in P$, we have $\omega_p(X_p^A) = A$, that is $\omega_p \circ i_p = \mathrm{id}_{T_eG}$.



b) $((\lhd\, g)^*\omega)|_p(X_p) = (\mathrm{Ad}_{g^{-1}})_*(\omega_p(X_p))$



c) ω *is a smooth one-form.*

Proof. a) Since $X_p^A \in V_pP$, by definition of ω we have

$$\omega_p(X_p^A) := i_p^{-1}(\mathrm{ver}(X_p^A)) = i_p^{-1}(X_p^A) = A.$$

b) First observe that the left hand side is linear in X_p. Consider the two cases

b.1) Suppose that $X_p \in V_pP$. Then $X_p = X_p^A$ for some $A \in T_eG$. Hence

$$\begin{aligned} ((\lhd\, g)^*\omega)|_p(X_p^A) &= \omega_{p\lhd g}((\lhd\, g)_* X_p^A) \\ &= \omega_{p\lhd g}\Big(X_{p\lhd g}^{(\mathrm{Ad}_{g^{-1}})_* A}\Big) \\ &= (\mathrm{Ad}_{g^{-1}})_* A \\ &= (\mathrm{Ad}_{g^{-1}})_*(\omega_p(X_p^A)) \end{aligned}$$

b.2) Suppose now that $X_p \in H_pP = \ker(\omega_p)$. Then

$$((\lhd\, g)^*\omega)|_p(X_p) = \omega_{p\lhd g}((\lhd\, g)_* X_p) = 0$$

since $(\lhd\, g)_* X_p \in H_{p\lhd g}P = \ker(\omega_{p\lhd g})$.

Let $X_p \in T_pP$. We have

$$\begin{aligned} ((\lhd\, g)^*\omega)|_p(X_p) &= ((\lhd\, g)^*\omega)|_p(\mathrm{ver}(X_p) + \mathrm{hor}(X_p)) \\ &= ((\lhd\, g)^*\omega)|_p(\mathrm{ver}(X_p)) + ((\lhd\, g)^*\omega)|_p(\mathrm{hor}(X_p)) \\ &= (\mathrm{Ad}_{g^{-1}})_*(\omega_p(\mathrm{ver}(X_p))) + 0 \\ &= (\mathrm{Ad}_{g^{-1}})_*(\omega_p(\mathrm{ver}(X_p))) + (\mathrm{Ad}_{g^{-1}})_*(\omega_p(\mathrm{hor}(X_p))) \\ &= (\mathrm{Ad}_{g^{-1}})_*(\omega_p(\mathrm{ver}(X_p) + \mathrm{hor}(X_p))) \\ &= (\mathrm{Ad}_{g^{-1}})_*(\omega_p(X_p)) \end{aligned}$$

c) We have $\omega = i^{-1} \circ \mathrm{ver}$ and both i^{-1} and ver are smooth. □

F Local representations of a connection on the base manifold: Yang-Mills fields

We have seen how to associate a connection one-form to a connection, i.e. a certain Lie-algebra-valued one-form to a smooth choice of horizontal spaces on the principal bundle. We will now study how we can express this connection one-form locally on the base manifold of the principal bundle.

F.1 Yang-Mills fields and local representations

Definition. Given a connection one-form ω on P, such a local section σ induces

i) a *Yang-Mills field* $\omega^U : \Gamma(TU) \xrightarrow{\sim} T_eG$ given by

$$\omega^U := \sigma^*\omega;$$

ii) a *local trivialisation* of the principal bundle P, i.e. a map

$$\begin{aligned} h\colon U \times G &\to P \\ (m, g) &\mapsto \sigma(m) \lhd g; \end{aligned}$$

iii) a *local representation* of ω on U by

$$h^*\omega\colon \Gamma(T(U \times G)) \xrightarrow{\sim} T_eG.$$

Note that, at each point $(m, g) \in U \times G$, we have

$$T_{(m,g)}(U \times G) \cong_{\text{Lie alg}} T_mU \oplus T_gG.$$

Remark .106. Both the Yang-Mills field ω^U and the local representation $h^*\omega$ encode the information carried by ω locally on U. Since $h^*\omega$ involves $U \times G$ while ω^U doesn't, one might guess that $h^*\omega$ gives a more "accurate" picture of ω on U than the Yang-Mills field. But in fact, this is not the case. They both contain the same amount of local information about the connection one-form ω.

F.2 The Maurer-Cartan form

Theorem .107. *For all $v \in T_mU$ and $\gamma \in T_gG$, we have*

$$(h^*\omega)_{(m,g)}(v, \gamma) = (\mathrm{Ad}_{g^{-1}})_*(\omega^U(v)) + \Xi_g(\gamma),$$

where Ξ_g is the Maurer-Cartan form

$$\begin{aligned} \Xi_g\colon T_gG &\xrightarrow{\sim} T_eG \\ L_g^A &\mapsto A. \end{aligned}$$

Example .108. Let $(\mathrm{GL}^+(d,\mathbb{R}), x)$ be a chart on $\mathrm{GL}(d,\mathbb{R})$, where $\mathrm{GL}^+(d,\mathbb{R})$ denotes an open subset of $\mathrm{GL}(d,\mathbb{R})$ containing the identity $\mathrm{id}_{\mathbb{R}^d}$, and let $x^i{}_j \colon \mathrm{GL}^+(d,\mathbb{R}) \to \mathbb{R}$ denote the corresponding co-ordinate functions

$$\begin{array}{ccc} \mathrm{GL}^+(d,\mathbb{R}) & \xrightarrow{\quad x \quad} & x(\mathrm{GL}^+(d,\mathbb{R})) \subseteq \mathbb{R}^{d^2} \\ & \searrow^{x^i{}_j} & \downarrow \mathrm{proj}^i{}_j \\ & & \mathbb{R} \end{array}$$

so that $x^i{}_j(g) := g^i{}_j$. Recall that the co-ordinate functions are smooth maps on the chart domain, i.e. we have $x^i{}_j \in \mathcal{C}^\infty(\mathrm{GL}^+(d,\mathbb{R}))$. Also recall that to each $A \in T_{\mathrm{id}_{\mathbb{R}^d}} \mathrm{GL}(d,\mathbb{R})$ there is associated a left-invariant vector field

$$L^A \colon \mathcal{C}^\infty(\mathrm{GL}^+(d,\mathbb{R})) \xrightarrow{\sim} \mathcal{C}^\infty(\mathrm{GL}^+(d,\mathbb{R}))$$

which, at each point $g \in \mathrm{GL}(d,\mathbb{R})$, is the tangent vector to the curve

$$\gamma^A(t) := g \bullet \exp(tA).$$

Consider the action of L^A on the co-ordinate functions:

$$\begin{aligned} (L^A x^i{}_j)(g) &= [x^i{}_j(g \bullet \exp(tA))]'(0) \\ &= [x^i{}_j(g \bullet \mathrm{e}^{tA})]'(0) \\ &= (g^i{}_k (\mathrm{e}^{tA})^k{}_j)'(0) \\ &= g^i{}_k A^k{}_j, \end{aligned}$$

where we have used the fact that for a matrix Lie group, the exponential map is just the ordinary exponential

$$\exp(A) = \mathrm{e}^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

Hence, we can write

$$L^A|_g = g^i{}_k A^k{}_j \left(\frac{\partial}{\partial x^i{}_j} \right)_g$$

from which we can read-off the Maurer-Cartan form of $\mathrm{GL}(d,\mathbb{R})$

$$(\Xi_g)^i{}_j := (g^{-1})^i{}_k (\mathrm{d}x^k{}_j)_g.$$

Indeed, we can quickly check that

$$\begin{aligned} (\Xi_g)^i{}_j(L^A) &= (g^{-1})^i{}_k (\mathrm{d}x^k{}_j)_g \left(g^p{}_r A^r{}_q \left(\frac{\partial}{\partial x^p{}_q} \right)_g \right) \\ &= (g^{-1})^i{}_k g^p{}_r A^r{}_q \delta^k_p \delta^q_j \\ &= (g^{-1})^i{}_p g^p{}_r A^r{}_j \\ &= \delta^i_r A^r{}_j \\ &= A^i{}_j. \end{aligned}$$

F.3 The gauge map

Definition. Within the above set-up, the *gauge map* is the map

$$\Omega\colon U_1 \cap U_2 \to G$$

where, for each $m \in U_1 \cap U_2$, the Lie group element $\Omega(m) \in G$ satisfies

$$\sigma_2(m) = \sigma_1(m) \triangleleft \Omega(m).$$

Note that since the G-action $\triangleleft$ on P is free, for each m there exists a unique $\Omega(m)$ satisfying the above condition, and hence the gauge map Ω is well-defined.

Theorem .109. *Under the above assumptions, we have*

$$(\omega^{U_2})_m = (\mathrm{Ad}_{\Omega^{-1}(m)})_*(\omega^{U_1}) + (\Omega^*\Xi_g)_m.$$

G Parallel transport

The idea of parallel transport on a principal bundle hinges on that of horizontal lift of a curve on the base manifold, which is a lifting to a curve on the principal bundle in the sense that the projection to the base manifold of this curve gives the curve we started with. In particular, if the principal bundle is equipped with a connection, we would like to impose some extra conditions on this lifting, so that it "connects" nearby fibres in a nice way.

G.1 Horizontal lifts to the principal bundle

Definition. Let (P, π, M) be a principal G-bundle equipped with a connection and let $\gamma\colon [0,1] \to M$ be a curve on M. The *horizontal lift of* γ *through* $p_0 \in P$ is the unique curve

$$\gamma^{\uparrow}\colon [0,1] \to P$$

with $\gamma^{\uparrow}(0) = p_0 \in \operatorname{preim}_\pi(\{\gamma(0)\})$ satisfying

i) $\pi \circ \gamma^{\uparrow} = \gamma$;

ii) $\forall\, \lambda \in [0,1]: \ \operatorname{ver}(X_{\gamma^{\uparrow},\gamma^{\uparrow}(\lambda)}) = 0$;

iii) $\forall\, \lambda \in [0,1]: \ \pi_*(X_{\gamma^{\uparrow},\gamma^{\uparrow}(\lambda)}) = X_{\gamma,\gamma(\lambda)}$.

We have the following result characterizing the curve g appearing above.

Theorem .110. *The (first order) ODE satisfied by the curve* $g\colon [0,1] \to G$ *is*

$$(\mathrm{Ad}_{g(\lambda)^{-1}})_*(\omega_{\delta(\lambda)}(X_{\delta,\delta(\lambda)})) + \Xi_{g(\lambda)}(X_{g,g(\lambda)}) = 0$$

with the initial condition $g(0) = g_0$.

Corollary .111. *If* G *is a matrix group, then the above ODE takes the form*

$$g(\lambda)^{-1}(\omega_{\delta(\lambda)}(X_{\delta,\delta(\lambda)}))g(\lambda) + g(\lambda)^{-1}\dot g(\lambda) = 0$$

where we denoted matrix multiplication by juxtaposition and $\dot g(\lambda)$ *denotes the derivative with respect to* λ *of the matrix entries of* g. *Equivalently, by multiplying both sides on the left by* $g(\lambda)$,

$$\dot g(\lambda) = -(\omega_{\delta(\lambda)}(X_{\delta,\delta(\lambda)}))g(\lambda).$$

G.2 Solution of the horizontal lift ODE by a path-ordered exponential

As a first step towards the solution of our ODE, consider

$$g(t) := g_0 - \int_0^t \mathrm{d}\lambda\, \Gamma_\mu(\gamma(\lambda))\, \dot\gamma^\mu(\lambda) g(\lambda).$$

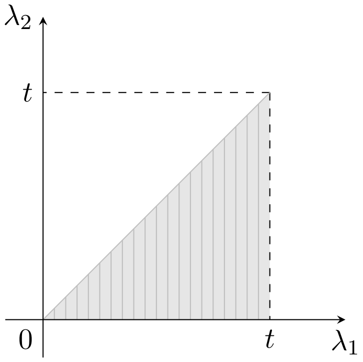
This doesn't seem to have brought us far since the function g that we would like to determine appears again on the right hand side. However, we can now iterate this definition to obtain

$$\begin{aligned} g(t) &= g_0 - \int_0^t \mathrm{d}\lambda_1\, \Gamma_\mu(\gamma(\lambda_1))\dot\gamma^\mu(\lambda_1)\Big(g_0 - \int_0^{\lambda_1} \mathrm{d}\lambda_2\, \Gamma_\nu(\gamma(\lambda_2))\dot\gamma^\nu(\lambda_2)\Big) \\ &= g_0 - \int_0^t \mathrm{d}\lambda_1\, \Gamma_\mu(\gamma(\lambda_1))\dot\gamma^\mu(\lambda_1) g_0 + \int_0^t \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2\, \Gamma_\mu(\gamma(\lambda_1))\dot\gamma^\mu(\lambda_1)\Gamma_\nu(\gamma(\lambda_2))\dot\gamma^\nu(\lambda_2) g(\lambda_2). \end{aligned}$$

Matters seem to only get worse, until one realises that the first integral no longer contains the unknown function g. Hence, the above expression provides a "first-order" approximation to g. It is clear that we can get higher-order approximations by iterating this process

$$\begin{aligned} g(t) = g_0 &- \int_0^t \mathrm{d}\lambda_1 \, \Gamma_\mu(\gamma(\lambda_1))\dot{\gamma}^\mu(\lambda_1) g_0 \\ &+ \int_0^t \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \, \Gamma_\mu(\gamma(\lambda_1))\dot{\gamma}^\mu(\lambda_1)\Gamma_\nu(\gamma(\lambda_2))\dot{\gamma}^\nu(\lambda_2) g_0 \\ &\vdots \\ &+ (-1)^n \int_0^t \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \cdots \int_0^{\lambda_{n-1}} \mathrm{d}\lambda_n \, \Gamma_\mu(\gamma(\lambda_1))\dot{\gamma}^\mu(\lambda_1) \cdots \Gamma_\nu(\gamma(\lambda_n))\dot{\gamma}^\nu(\lambda_n) \, g(\lambda_n). \end{aligned}$$

Note how the range of each integral depends on the integration variable of the previous integral. It would much nicer if we could have the same range in each integral. In fact, there is a standard trick to achieve this. The region of integration in the double integral is



and if the integrand $f(\lambda_1, \lambda_2)$ is invariant under the exchange $\lambda_1 \leftrightarrow \lambda_2$, we have

$$\int_0^t \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \, f(\lambda_1, \lambda_2) = \frac{1}{2} \int_0^t \mathrm{d}\lambda_1 \int_0^t \mathrm{d}\lambda_2 \, f(\lambda_1, \lambda_2).$$

Generalising to n dimensions, we have

$$\int_0^t \mathrm{d}\lambda_1 \cdots \int_0^{\lambda_{n-1}} \mathrm{d}\lambda_n \, f(\lambda_1, \ldots, \lambda_n) = \frac{1}{n!} \int_0^t \mathrm{d}\lambda_1 \cdots \int_0^t \mathrm{d}\lambda_n \, f(\lambda_1, \ldots, \lambda_n)$$

if f is invariant under any permutation of its arguments. Moreover, since each term in our integrands only depends on one integration variable at a time, we can use

$$\int_0^t \mathrm{d}\lambda_1 \cdots \int_0^t \mathrm{d}\lambda_n \, f_1(\lambda_1) \cdots f_n(\lambda_n) = \left(\int_0^t \mathrm{d}\lambda_1 \, f(\lambda_1) \right) \cdots \left(\int_0^t \mathrm{d}\lambda_n \, f(\lambda_n) \right)$$

so that, in our case, we would have

$$\begin{aligned} g(t) &= \left(\sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(\int_0^t \mathrm{d}\lambda \, \Gamma_\mu(\gamma(\lambda))\dot{\gamma}^\mu(\lambda) \right)^n \right) g_0 \\ &= \exp\left(- \int_0^t \mathrm{d}\lambda \, \Gamma_\mu(\gamma(\lambda))\dot{\gamma}^\mu(\lambda) \right) g_0. \end{aligned}$$

However, our integrands are Lie-algebra-valued (that is, matrix valued), and since the factors therein need not commute, they are not invariant under permutations of the independent variables. Hence, the above formula doesn't work. Instead, we write

$$g(t) = \mathrm{P}\exp\Big(-\int_0^t \mathrm{d}\lambda\, \Gamma_\mu(\gamma(\lambda))\dot\gamma^\mu(\lambda)\Big) g_0,$$

where the *path-ordered exponential* $\mathrm{P}\exp$ is defined to yield the correct expression for $g(t)$.

Summarising, we have the following.

Proposition .112. *For a principal G-bundle (P, π, M), where G is a matrix Lie group, the horizontal lift of a curve $\gamma\colon [0,1] \to U$ through $p_p \in \mathrm{preim}_\pi(\{U\})$, where (U, x) is a chart on M, is given in terms of a local section $\sigma\colon U \to P$ by the explicit expression*

$$\gamma^\uparrow(\lambda) = (\sigma \circ \gamma)(\lambda) \lhd \Big(\mathrm{P}\exp\Big(-\int_0^\lambda \mathrm{d}\widetilde\lambda\, \Gamma_\mu(\gamma(\widetilde\lambda))\dot\gamma^\mu(\widetilde\lambda)\Big) g_0\Big).$$

Definition. Let $\gamma_p^\uparrow\colon [0,1] \to P$ be the horizontal lift through $p \in \mathrm{preim}_\pi(\{\gamma(0)\})$ of the curve $\gamma\colon [0,1] \to M$. The *parallel transport map along* γ is the map

$$\begin{aligned} T_\gamma\colon \mathrm{preim}_\pi(\{\gamma(0)\}) &\to \mathrm{preim}_\pi(\{\gamma(1)\}) \\ p &\mapsto \gamma_p^\uparrow(1). \end{aligned}$$

Loops and holonomy groups

Consider the case of loops, i.e. curves $\gamma\colon [0,1] \to M$ for which $\gamma(0) = \gamma(1)$. Fix some $p \in \mathrm{preim}_\pi(\{\gamma(0)\})$. The condition that $\pi \circ \gamma_p^\uparrow = \gamma$ then implies that $\gamma_p^\uparrow(0)$ and $\gamma_p^\uparrow(1)$ belong to the same fibre. Hence, there exists a unique $g_\gamma \in G$ such that

$$\gamma_p^\uparrow(1) = \gamma_p^\uparrow(0) \lhd g_\gamma = p \lhd g_\gamma.$$

Definition. Let ω be a connection one-form on the principal G-bundle (P, π, M). Let $\gamma\colon [0,1] \to M$ be a loop with base-point $a \in M$, i.e. $\gamma(0) = \gamma(1) = a$. The subgroup of G

$$\mathrm{Hol}_a(\omega) := \{g_\gamma \mid \gamma_p^\uparrow(1) = p \lhd g_\gamma \text{ for some loop } \gamma\}$$

is called the *holonomy group* of ω on P at the base-point a.

G.3 Horizontal lifts to the associated bundle

Definition. Let (P, π, M) be a principal G-bundle and ω a connection one-form on P. Let (P_F, π_F, M) be an associated fibre bundle of P on whose typical fibre F the Lie group G acts on the left by $\rhd$. Let $\gamma\colon [0,1] \to M$ be a curve on M and let $\gamma_p^\uparrow$ be its horizontal lift to P through $p \in \mathrm{preim}_\pi(\{\gamma(0)\})$. Then the *horizontal lift* of γ to the associated bundle P_F through the point $[p, f] \in P_F$ is the curve

$$\begin{aligned} \overset{P_F}{\gamma}{}^\uparrow_{[p,f]}\colon [0,1] &\to P_F \\ \lambda &\mapsto [\gamma_p^\uparrow(\lambda), f] \end{aligned}$$

For instance, we have the obvious parallel transport map.

Definition. The *parallel transport map* on the associated bundle is given by

$$\begin{aligned} T_\gamma^{P_F} \colon \mathrm{preim}_{\pi_F}(\{\gamma(0)\}) &\to \mathrm{preim}_{\pi_F}(\{\gamma(1)\}) \\ [p,f] &\mapsto \overset{P_F}{\underset{[p,f]}{\gamma^{\uparrow}}}(1). \end{aligned}$$

Remark .113. If F is a vector space and $\rhd \colon G \times F \to F$ is fibre-wise linear, i.e. for each fixed $g \in G$, the map $(g \rhd -)\colon F \to F$ is linear, then (P_F, π_F, M) is called a *vector bundle*. The basic idea of a covariant derivative is as follows. Let $\sigma \colon U \to P_F$ be a local section of the associated bundle. We would like to define the derivative of σ at the point $m \in U \subseteq M$ in the direction $X \in T_m M$. By definition, there exists a curve $\gamma \colon (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = m$ such that $X = X_{\gamma,m}$. Then for any $0 \leq t < \varepsilon$, the points $\overset{P_F}{\underset{[\sigma(m),f]}{\gamma^{\uparrow}}}(t)$ and $\sigma(\gamma(t))$ lie in the same fibre of P_F. But since the fibres are vector spaces, we can write the differential quotient

$$\frac{\sigma(\gamma(t)) - \overset{P_F}{\underset{[\sigma(m),f]}{\gamma^{\uparrow}}}(t)}{t},$$

where the minus sign denotes the additive inverse in the vector space $\mathrm{preim}_{\pi_F}(\{\gamma(t)\})$ and hence define the derivative of σ at the point m in the direction X, or the derivative of σ along γ at $\gamma(0) = m$, by

$$\lim_{t \to 0} \frac{\sigma(\gamma(t)) - \overset{P_F}{\underset{[\sigma(m),f]}{\gamma^{\uparrow}}}(t)}{t}$$

(of course, this makes sense as soon as we have a topology on the fibres). We will soon present a more abstract approach.

H Curvature and torsion on principal bundles

In more elementary treatments of differential geometry or general relativity, curvature and torsion are mentioned together as properties of a covariant derivative over the tangent or the frame bundle. However, this is not the case. Torsion requires additional structure beyond that induced by a connection.

H.1 Covariant exterior derivative and curvature

Definition. Let (P, π, M) be a principal G-bundle with connection one-form ω. Let ϕ be a k-form (i.e. an anti-symmetric, $\mathcal{C}^\infty(P)$-multilinear map) with values in some module V. Then then *exterior covariant derivative* of ϕ is

$$\begin{aligned} \mathrm{D}\phi\colon \quad & \Gamma(TP)^{\times(k+1)} \to V \\ & (X_1, \dots, X_{k+1}) \mapsto \mathrm{d}\phi(\mathrm{hor}(X_1), \dots, \mathrm{hor}(X_{k+1})). \end{aligned}$$

Definition. Let (P, π, M) be a principal G-bundle with connection one-form ω. The *curvature* of the connection one-form ω is the Lie-algebra-valued 2-form on P

$$\Omega\colon \Gamma(TP) \times \Gamma(TP) \to T_eG$$

defined by

$$\Omega := \mathrm{D}\omega.$$

For calculation purposes, we would like to make this definition a bit more explicit.

Proposition .114. *Let ω be a connection one-form and Ω its curvature. Then*

$$\Omega = \mathrm{d}\omega + \omega \mathbin{\text{⩓}} \omega \tag{$\star$}$$

with the second term on the right hand side defined as

$$(\omega \mathbin{\text{⩓}} \omega)(X, Y) := [\![\omega(X), \omega(Y)]\!]$$

where $X, Y \in \Gamma(TP)$ and the double bracket denotes the Lie bracket on T_eG.

Remark .115. If G is a matrix Lie group, and hence T_eG is an algebra of matrices of the same size as those of G, then we can write

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j.$$

Proof. Since Ω is $\mathcal{C}^\infty$-bilinear, it suffices to consider the following three cases.

a) Suppose that $X, Y \in \Gamma(TP)$ are both vertical, that is, there exist $A, B \in T_eG$ such that $X = X^A$ and $Y = X^B$. Then the left hand side of our equation reads

$$\begin{aligned} \Omega(X^A, X^B) &:= \mathrm{D}\omega(X^A, X^B) \\ &= \mathrm{d}\omega(\mathrm{hor}(X^A), \mathrm{hor}(X^B)) \\ &= \mathrm{d}\omega(0, 0) \\ &= 0 \end{aligned}$$

while the right hand side is

$$\begin{aligned}
d\omega(X^A, X^B) + (\omega \owedge \omega)(X^A, X^B) &= X^A(\omega(X^B)) - X^B(\omega(X^A)) \\
&\quad - \omega([X^A, X^B]) + [\![\omega(X^A), \omega(X^B)]\!] \\
&= X^A(B) - X^B(A) \\
&\quad - \omega(X^{[\![A,B]\!]}) + [\![A, B]\!] \\
&= -[\![A, B]\!] + [\![A, B]\!] \\
&= 0.
\end{aligned}$$

Note that we have used the fact that the map

$$\begin{aligned}
i\colon\ T_eG &\to \Gamma(TP) \\
A &\mapsto X^A
\end{aligned}$$

is a Lie algebra homomorphism, and hence

$$X^{[\![A,B]\!]} = i([\![A, B]\!]) = [i(A), i(B)] = [X^A, X^B],$$

where the single square brackets denote the Lie bracket on $\Gamma(TP)$.

b) Suppose that $X, Y \in \Gamma(TP)$ are both horizontal. Then we have

$$\Omega(X, Y) := D\omega(X, Y) = d\omega(\mathrm{hor}(X), \mathrm{hor}(Y)) = d\omega(X, Y)$$

and

$$(\omega \owedge \omega)(X, Y) = [\![\omega(X), \omega(Y)]\!] = [\![0, 0]\!] = 0.$$

Hence the equation holds in this case.

c) W.l.o.g suppose that $X \in \Gamma(TP)$ is horizontal while $Y = X^A \in \Gamma(TP)$ is vertical. Then the left hand side is

$$\Omega(X, X^A) := D\omega(X, X^A) = d\omega(\mathrm{hor}(X), \mathrm{hor}(X^A)) = d\omega(\mathrm{hor}(X), 0) = 0.$$

while the right hand side gives

$$\begin{aligned}
d\omega(X, X^A) + (\omega \owedge \omega)(X, X^A) &= X(\omega(X^A)) - X^A(\omega(X)) \\
&\quad - \omega([X, X^A]) + [\![\omega(X), \omega(X^A)]\!] \\
&= X(A) - X^A(0) \\
&\quad - \omega(X^{[\![A,B]\!]}) + [\![0, A]\!] \\
&= -\omega([X, X^A]) \\
&= 0,
\end{aligned}$$

where the only non-trivial step, which is left as an exercise, is to show that if X is horizontal and Y is vertical, then $[X, Y]$ is again horizontal. □

Definition. Let (P, π, M) be a principal G-bundle and let Ω be the curvature associated to a connection one-form on P. Let $\sigma \in \Gamma(TU)$ be a local section on M. Then, the two-form

$$\mathrm{Riem} \equiv F := \sigma^*\Omega \in \Omega^2(U) \otimes T_eG$$

is called the *Yang-Mills field strength*.

Theorem .116 (First Bianchi identity)**.** *Let Ω be the curvature two-form associated to a connection one-form ω on a principal bundle. Then*

$$\mathrm{D}\Omega = 0.$$

H.2 Torsion

Definition. Let (P, π, M) be a principal G-bundle and let V be the representation space of a linear $(\dim M)$-dimensional representation of the Lie group G. A *solder(ing) form* on P is a one-form $\theta \in \Omega^1(P) \otimes V$ such that

(i) $\forall\, X \in \Gamma(TP): \;\; \theta(\mathrm{ver}(X)) = 0$;

(ii) $\forall\, g \in G: \;\; g \triangleright ((\triangleleft\, g)^*\theta) = \theta$;

(iii) TM and P_V are isomorphic as associated bundles.

Definition. Let (P, π, M) be a principal G-bundle with connection one-form ω and let $\theta \in \Omega^1(P) \otimes V$ be a solder form on P. Then

$$\Theta := \mathrm{D}\theta \in \Omega^2(P) \otimes V$$

is the *torsion* of ω with respect to θ.

Theorem .117 (Second Bianchi identity)**.** *Let Θ be the torsion of a connection one-form ω with respect to a solder form θ on a principal bundle. Then*

$$\mathrm{D}\Theta = \Omega \mathbin{\curlywedge} \theta.$$

I Covariant derivatives

There is a neater approach to covariant differentiation, which will now discuss.

I.1 Equivalence of local sections and equivariant functions

Theorem .118. *Let (P, π, M) be a principal G-bundle and (P_F, π_F, M) be an associated bundle. Let (U, x) be a chart on M. The local sections $\sigma\colon U \to P_F$ are in bijective correspondence with G-equivariant functions $\phi\colon \operatorname{preim}_\pi(U) \subseteq P \to F$, where the G-equivariance condition is*

$$\forall\, g \in G : \forall\, p \in \operatorname{preim}_\pi(U) : \;\; \phi(g \lhd p) = g^{-1} \rhd \phi(p).$$

Proof. (a) Let $\phi \operatorname{preim}_\pi(U) \to F$ be G-equivariant. Define

$$\begin{aligned} \sigma_\phi\colon U &\to P_F \\ m &\mapsto [p, \phi(p)] \end{aligned}$$

where p is any point in $\operatorname{preim}_\pi(\{m\})$. First, we should check that σ_ϕ is well-defined. Let $p, \widetilde{p} \in \operatorname{preim}_\pi(\{m\})$. Then, there exists a unique $g \in G$ such that $\widetilde{p} = p \lhd g$. Then, by the G-equivariance of ϕ, we have

$$[\widetilde{p}, \phi(\widetilde{p})] = [p \lhd g, \phi(p \lhd g)] = [p \lhd g, g^{-1} \rhd \phi(p)] = [p, \phi(p)]$$

and hence, σ_ϕ is well-defined. Moreover, since for all $g \in G$

$$\pi_F([p, \phi(p)]) = \pi(p) = \pi(p \lhd g) = \pi_F([p \lhd g, g^{-1} \rhd \phi(p)]),$$

we have $\pi_F \circ \sigma_\phi = \mathrm{id}_U$ and thus, σ_ϕ is a local section.

(b) Let $\sigma\colon U \to P_F$ be a local section. Define

$$\begin{aligned} \phi_\sigma\colon \operatorname{preim}_\pi(U) &\to F \\ p &\mapsto i_p^{-1}(\sigma(\pi(p))) \end{aligned}$$

where i_p^{-1} is the inverse of the map

$$\begin{aligned} i_p\colon F &\to \operatorname{preim}_{\pi_F}(\{\pi(p)\}) \subseteq P_F \\ f &\mapsto [p, f]. \end{aligned}$$

Observe that, for all $g \in G$, we have

$$i_p(f) := [p, f] = [p \lhd g, g^{-1} \rhd f] =: i_{p \lhd g}(g^{-1} \rhd f).$$

Let us now show that ϕ_σ is G-equivariant. We have

$$\begin{aligned} \phi_\sigma(p \lhd g) &= i_{p\lhd g}^{-1}(\sigma(\pi(p \lhd g))) \\ &= i_{p\lhd g}^{-1}(\sigma(\pi(p))) \\ &= i_{p\lhd g}^{-1}(i_p(\phi_\sigma(p))) \\ &= i_{p\lhd g}^{-1}(i_{p\lhd g}(g^{-1} \rhd \phi_\sigma(p))) \\ &= g^{-1} \rhd \phi_\sigma(p), \end{aligned}$$

which is what we wanted.

(c) We now show that these constructions are the inverses of each other, i.e.

$$\sigma_{\phi_\sigma} = \sigma, \qquad \phi_{\sigma_\phi} = \phi.$$

Let $m \in U$. Then, we have

$$\begin{aligned}
\sigma_{\phi_\sigma}(m) &= [p, \phi_\sigma(p)] \\
&= [p, i_p^{-1}(\sigma(\pi(p)))] \\
&= i_p(i_p^{-1}(\sigma(\pi(p)))) \\
&= \sigma(\pi(p)) \\
&= \sigma(m)
\end{aligned}$$

and hence $\sigma_{\phi_\sigma} = \sigma$. Now let $p \in \mathrm{preim}_\pi(U)$. Then, we have

$$\begin{aligned}
\phi_{\sigma_\phi}(p) &= i_p^{-1}(\sigma_\phi(\pi(p))) \\
&= i_p^{-1}([p, \phi(p)]) \\
&= i_p^{-1}(i_p(\phi(p))) \\
&= \phi(p)
\end{aligned}$$

and hence, $\phi_{\sigma_\phi} = \phi$. □

I.2 Linear actions on associated vector fibre bundles

We now specialise to the case where F is a vector space, and hence we can require the left action $G \triangleright \colon F \xrightarrow{\sim} F$ to be linear.

Proposition .119. *Let (P, π, M) be a principal G-bundle, and let (P_F, π_F, M) be an associated bundle, where G is a matrix Lie group, F is a vector space, and the left G-action on F is linear. Let $\phi \colon P \to F$ be G-equivariant. Then*

$$\phi(p \triangleleft \exp(At)) = \exp(-At) \triangleright \phi(p),$$

where $p \in P$ and $A \in T_eG$.

Corollary .120. *With the same assumptions as above, let $A \in T_eG$ and let ω be a connection one-form on (P, π, M). Then*

$$\mathrm{d}\phi(X^A) + \omega(X^A) \triangleright \phi = 0.$$

Proof. Since ϕ is G-equivariant, by applying the previous proposition, we have

$$\phi(p \triangleleft \exp(At)) = \exp(-At) \triangleright \phi(p)$$

for any $p \in P$. Hence, differentiating with respect to t yields

$$\begin{aligned}
(\phi(p \triangleleft \exp(At)))'(0) &= (\exp(-At) \triangleright \phi(p))'(0) \\
\mathrm{d}_p\phi(X^A) &= -A \triangleright \phi(p) \\
\mathrm{d}_p\phi(X^A) &= -\omega(X^A) \triangleright \phi(p)
\end{aligned}$$

for all $p \in P$ and hence, the claim holds. □

I.3 Construction of the covariant derivative

With a covariant derivative, i.e. an "operator" ∇ defined previously, ee now have the following result.

Proposition .121. *Let $\phi\colon P \to F$ be G-equivariant and let $X \in T_pP$. Then*

$$\mathrm{D}\phi(X) = \mathrm{d}\phi(X) + \omega(X) \triangleright \phi$$

Proof. (a) Suppose that X is vertical, that is, $X = X^A$ for some $A \in T_eG$. Then,

$$\mathrm{D}\phi(X) = \mathrm{d}\phi(\mathrm{hor}(X)) = 0$$

and

$$\mathrm{d}\phi(X^A) + \omega(X^A) \triangleright \phi = 0$$

by the previous corollary.

(b) Suppose that X is horizontal. Then,

$$\mathrm{D}\phi(X) = \mathrm{d}\phi(X)$$

and $\omega(X) = 0$, so that we have

$$\mathrm{D}\phi(X) = \mathrm{d}\phi(X) + \omega(X) \triangleright \phi. \qquad \square$$

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